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IMPROVEMENT IN TARGET DETECTION WITH A FREQUENCY AGILE PULSED
NON-COHERENT RADAR

by

KENNETH LEE HORN

A DISSERTATION

Presented to the Faculty of the Graduate School of the
UNIVERSITY OF MISSOURI - ROLLA

In Partial Fulfillment of the Requirements for the Degree
DOCTOR OF PHILOSOPHY

in

ELECTRICAL ENGINEERING

1969

Harold D. Wallace
Advisor

Samuel J. Dyer, Jr.

J. R. Bett

J. J. Skitek

Frank J. Kern

Lee J. Bain

ABSTRACT

Frequency agile tracking radars have received considerable attention in the literature. Frequency agility is useful in reducing target scintillations, thereby reducing tracking errors under certain conditions. However, little attention has been devoted to the improvement that can be expected in target detection with a frequency agile radar. In the past, complete pulse-to-pulse decorrelation of the target returns has been assumed for all of the analyses concerning target detection with frequency agile radars. This assumption always implies a transmitter with a large bandwidth.

This dissertation analyses a pulsed non-coherent frequency agile radar applied to Swerling's Case I target. The target model is modified by assuming time decorrelation of the returns in the latter portion of the dissertation. Partial correlation of the returns is also allowed. It is shown that random selection of the transmitted frequency from a continuous probability density function yields a significant reduction in the transmitter bandwidth requirement, when compared to previous analyses, with a negligible decrease in detection range.

Two practical density functions for the transmitted frequency are studied in detail, the uniform and anti-sine distribution. However, the results are general in that the curves for detection probabilities may be applied directly to any pulsed non-coherent frequency agile radar whose transmitted frequency is obtained from a probability density function. The analysis yields a closed form solution, previously unknown, for detection probabilities in the case of Swerling's Case I target. A computer program is included which calculates detection

probabilities for Swerling's Case I target modified by a time decorrelation function.

ACKNOWLEDGEMENTS

The author would like to acknowledge the help and guidance of Dr. N.D. Wallace in the preparation of this dissertation. Acknowledgement is also due Dr. J.R. Betten for introducing the author to the field of signal processing, and guiding him throughout his graduate education.

The author is also appreciative of the help and cooperation of people too numerous to mention who assisted with the typing and preparation of this manuscript.

Finally, the author would like to acknowledge the sacrifices and patience of his family during the years of his graduate education.

TABLE OF CONTENTS

	Page
ABSTRACT	ii
ACKNOWLEDGEMENTS	iv
LIST OF FIGURES	vi
LIST OF TABLES	viii
LIST OF SYMBOLS	ix
I. INTRODUCTION TO PULSED RADAR AND THE CONCEPT OF FREQUENCY AGILITY	1
II. SEARCH RADAR WITH FREQUENCY AGILITY	15
III. CALCULATION OF THE CORRELATION COEFFICIENT	23
IV. FREQUENCY AGILE RADAR DETECTION PROBABILITIES FOR SWERLING'S CASE I TARGET	34
V. DETECTION PROBABILITIES FOR SWERLING'S MODIFIED CASE I TARGET	48
VI. CONCLUSIONS AND SUGGESTIONS FOR FURTHER INVESTIGATIONS	60
BIBLIOGRAPHY	62
APPENDIX A - DERIVATION OF EQUATION 7	65
APPENDIX B - DERIVATION OF EQUATION 25	70
APPENDIX C - DERIVATION OF EQUATION 68	73
APPENDIX D - COMPUTER PROGRAM	77
VITA	82

LIST OF FIGURES

Figure	Page
1. Block diagram of receiver	6
2. Probability of detection vs. R/R_0 for $N=10$ and 100 , Swerling's Case I and II, probability of false alarm = 10^{-6}	14
3. Illustration of target depth, D	15
4. Illustration of random sampling from a sine wave	27
5. Illustration of sampling from a sine wave with unknown starting phase	29
6. Correlation coefficient vs. the product of target depth, D , in meters, and transmitter bandwidth, Δf , in megahertz	32
7. Probability of detection vs. R/R_0 for $N=10$, probability of false alarm = 10^{-6}	37
8. Probability of detection vs. R/R_0 for $N=10$, probability of false alarm = 10^{-8}	38
9. Probability of detection vs. R/R_0 for $N=10$, probability of false alarm = 10^{-10}	39
10. Probability of detection vs. R/R_0 for $N=25$, probability of false alarm = 10^{-6}	40
11. Probability of detection vs. R/R_0 for $N=25$, probability of false alarm = 10^{-8}	41
12. Probability of detection vs. R/R_0 for $N=25$, probability of false alarm = 10^{-10}	42
13. Probability of detection vs. R/R_0 for $N=100$, probability of false alarm = 10^{-6}	43
14. Probability of detection vs. R/R_0 for $N=100$, probability of false alarm = 10^{-8}	44
15. Probability of detection vs. R/R_0 for $N=100$, probability of false alarm = 10^{-10}	45
16. Probability of detection vs. R/R_0 for $N=10$, probability of false alarm = 10^{-6} , product of τ_c and $P_{rr} = 10$	52

Figure	Page
17. Probability of detection vs. R/R_0 for $N=10$, probability of false alarm $=10^{-6}$, product of τ_c and $P_{rr} = 25$	53
18. Probability of detection vs. R/R_0 for $N=10$, probability of false alarm $=10^{-6}$, product of τ_c and $P_{rr} = 100$	54
19. Probability of detection vs. R/R_0 for $N=10$, probability of false alarm $=10^{-10}$, product of τ_c and $P_{rr} = 10$	55
20. Probability of detection vs. R/R_0 for $N=10$, probability of false alarm $=10^{-10}$, product of τ_c and $P_{rr} = 25$	56
21. Probability of detection vs. R/R_0 for $N=10$, probability of false alarm $=10^{-10}$, product of τ_c and $P_{rr} = 100$	57
22. Probability of detection vs. R/R_0 for $N=25$, probability of false alarm $=10^{-6}$, correlation coefficient = 1 in all cases	58
23. Probability of detection vs. R/R_0 for $N=25$, probability of false alarm $=10^{-10}$, correlation coefficient = 1 in all cases	59

LIST OF TABLES

Table	Page
I. Bias level for $N = 10, 25$, and 100 , probability of false alarm = 10^{-6} , 10^{-8} , and 10^{-10}	46

LIST OF SYMBOLS

A_e	- effective area of receiving antenna
B	- bandwidth of IF stage
c	- velocity of light, 3×10^8 meters/second
D	- target depth in meters
f_o	- transmitter center frequency
f_i	- frequency of the i th transmitter pulse
G	- gain of transmitting antenna
k	- Boltzmann's constant
L	- loss factor in range equation
N	- number of pulses integrated
\overline{NF}	- noise figure of the receiver
p	- Laplace transform variable
P_d	- probability of detection
P_f	- probability of false alarm
P_r	- power received at antenna
P_t	- transmitter power
P_{rr}	- pulse repetition rate
PDF-	probability density function
Q	- absolute value of f_i minus f_j
R	- range from antenna to target
R_o	- normalizing range parameter
S_{cs}	- target scattering cross section
T_r	- absolute temperature of the receiver
t_s	- sampling time for the anti-sine distribution
W	- $2\pi D \Delta f/c$

- x - input signal-to-noise power ratio
- \bar{x} - average input signal-to-noise power ratio
- x_i - input signal-to-noise power ratio for the i th pulse
- Y_b - bias level
- z - output of linear integrator

- Δf - transmitter bandwidth
- Δ_k - determinant of the covariance matrix of the voltage returns
- Φ - covariance matrix of the voltage returns
- ψ_0 - average noise power at the filter output
- σ^2 - average noise power per cycle, $k T_r \overline{NF}$, at the filter input (-204 dBW at room temperature)
- ρ - average correlation coefficient for the voltage returns
- τ_c - correlation time constant
- μ_i - i th eigenvalue of the normalized covariance matrix, or correlation matrix

CHAPTER I

INTRODUCTION TO PULSED RADAR AND THE CONCEPT OF FREQUENCY AGILITY

A. Introduction

The concept of frequency agile radar transmitters began with the investigations of Birkemeier and Wallace before 1963.⁽¹⁾ They were concerned with reducing the scintillation (glint and fading) in tracking radars. (Glint denotes the wandering of the apparent center of reflectivity of a target, while fading is the term used to describe fluctuations in the received echo signal strength). The technique commonly known as frequency agility consists of changing the frequency of the radar transmitter between pulses in a manner that reduces scintillations and improves the target detection probability in some instances. The technique is also useful from an ECCM viewpoint, because it is more difficult to jam a radar in which the transmitter frequency varies with time.

Birkemeier and Wallace analyzed a tracking radar which incorporated a fixed frequency change between each transmitter pulse. The frequency of each pulse in the sequence was assumed to be monotonically increasing (or decreasing) for a length of time at least as great as that required for the target returns to become completely decorrelated. The frequency change between pulses was sufficient for pulse-to-pulse decorrelation. The sequence could then be recycled if desired, to limit the transmitter bandwidth requirement. For long correlation times, practical sized targets, and high pulse repetition rates, this scheme calls for a transmitter possessing a large bandwidth. Lind has recently extended this analysis for

frequency agility when the transmitter frequency can be described only by a probability density function (hereinafter abbreviated to PDF).⁽²⁾ However, his analysis is not exact and the resulting estimate of transmitter bandwidth is also very large.

The application of frequency agility in search radars has received very little attention in the open literature. Perhaps the chief reason for this is that detection probabilities may be obtained directly from Swerling's work if the pulse-to-pulse frequency change is sufficient for complete decorrelation.⁽³⁾ Ray has published some limited calculations for the target detection case.⁽⁴⁾ He does not describe his system in detail, and makes vague generalizations which are obvious after studying the works of Schwartz⁽⁵⁾ and Swerling⁽³⁾. Schwartz has given the closed form solution for detection probabilities in the case where only two pulses are integrated, with arbitrary correlation between the two target returns. He does not relate these results to frequency agility because his work preceeded the concept of frequency agility. In a later work, Swerling has made a few calculations of the target detection probability for a frequency agile system.⁽⁶⁾ However, he also assumes the transmitter frequency is shifted a known amount. The transmitter radiates groups of pulses, with each group having the same number pulses (which may be one) and each pulse in the group having the same frequency. He assumes the frequency shift occurs only between groups, and is sufficient for complete decorrelation between groups. This does not reduce the transmitter bandwidth requirement appreciably. For a target whose return fluctuations fall in a class such that the fluctuations of the sum of all of

the pulses in a train of echos can be described by a chi-square distribution, the detection probabilities may be calculated by Swerling's analysis. He does not treat the case where the transmitter frequency can only be described by a PDF. Therefore, his analysis leads to an excessive estimate of transmitter bandwidth just as all other analyses known by the author at present.

B. Statement of the Problem

This dissertation calculates the probability of detection (P_d) for certain classes of targets illuminated by practical transmitters whose frequency can be described only through a PDF. The transmitter bandwidth requirement will be shown to be much less than that estimated for frequency agile radars with a fixed frequency difference between pulses. The radar under consideration is a standard non-coherent receiver consisting of a linear pre-detection stage, square law detector, linear post-detection integrator, and a threshold decision device. The transmitter power and far field pattern is assumed to be constant over the frequency band of interest. Clutter effects are ignored, and the only noise present is that due to the receiver, which is assumed to be white Gaussian noise. This is a realistic assumption for most ground-to-air or high altitude air-to-air search radars when no chaff or precipitation are present.

The fluctuations in the target returns are assumed to follow Swerling's Case I model.⁽³⁾ They are described by (1), and are presumed to be independent from scan-to-scan but completely correlated during any one scan (time on target). Later, (1) will be modified to take into account decorrelation of the returns during a scan time.

These are the models most usually applied to modern jet aircraft, and have been experimentally confirmed.^(7,8)

$$f(x|\bar{x}) = \exp(-x / \bar{x}) / \bar{x} \quad x \geq 0 \quad (1)$$

x = input signal-to-noise power ratio

\bar{x} = average of x over all target fluctuations

$f(a|b)$ is the well known conditional probability notation defined as f of a , given b .

C. Review of the Literature

The application of probability theory to the detection of radar targets probably first occurred during World War II. This is hinted at in Volume 24 of the Radiation Laboratory Series.⁽⁹⁾ One of the first semi-declassified publications was completed by Marcum in 1947.⁽¹⁰⁾ In this report, he assumed the target returns were non-fluctuating. At each range, a given target produced a fixed power return which followed the range equation, (2). The radar he considered was a pulsed, non-coherent search radar of the same type considered in this dissertation.

$$P_r = \frac{P_t G A_e S_{CS} L}{(4\pi R^2)^2} \quad (2)$$

P_r = power received at antenna

R = range from antenna to target

P_t = transmitter power

G = gain of transmitting antenna

A_e = effective area of receiving antenna

S_{CS} = scattering cross section of target

L = loss factor

The noise at the input to the IF stage of the receiver was assumed to be additive white Gaussian noise and the effective noise power per unit cycle was defined as $kT_r\bar{N}F$. k is Boltzmann's constant, T_r is the absolute temperature of the receiver, and $\bar{N}F$ is the noise figure of the receiver. Marcum then defined a normalized range R_o which is given by (3).

$$R_o = \left[\frac{P_t G A_e S_{CS} L}{16\pi^2 k T_r \bar{N}F B} \right]^{1/4} \quad (3)$$

B is the bandwidth of the IF stage, and all other terms were previously defined. Solving (2) and (3) for the ratio R/R_o yields (4).

$$R/R_o = (k T_r \bar{N}F B/P_r)^{1/4} \quad (4)$$

The average noise power at the filter output, as derived in Appendix A, is $k T_r \bar{N}F B$. By defining x as the ratio of received signal power per pulse to average noise power at the filter output, (4) becomes (5). The use of the range parameter R/R_o instead of x condenses the curves for P_d and is a standard parameter utilized to present the curves of P_d .

$$R/R_o = (1/x)^{1/4} \quad (5)$$

These equations are modified slightly if the filter bandwidth is not the reciprocal of the received pulse length. However, this reciprocal relationship is assumed throughout this dissertation for the decrease in signal-to-noise ratio is negligible for this filter when compared to that of the optimum matched filter. Any mismatch of filter band-

width and pulse length may be accounted for by modifying the parameter L in (2) and (3).

The receiver considered by Marcum is illustrated in Figure 1. Although this is not the optimum receiver, it is approximately optimum for small signal-to-noise ratios. The optimum receiver is identical with that of Figure 1 with the square law envelope detector replaced by a $\ln [I_0(\cdot)]$ envelope detector. ^(10,11) (\ln is the natural logarithm and I_0 is the modified Bessel function of the first kind, order zero). In addition, the square law detector is the only detector for which closed form solutions may be attained with any degree of ease. ^(10,12) The operation of the receiver is as follows: At a given fixed time after the transmitter pulse (corresponding to a fixed range), a range gate opens and a signal is presented to the integrator for a period of time at least as long (usually longer) as the received pulse length. After repeating this procedure N times, a decision is made by the decision device as to whether a target is present or not, by comparing the output of the integrator to a pre-determined threshold voltage. The integrator is then dumped and the process starts again.

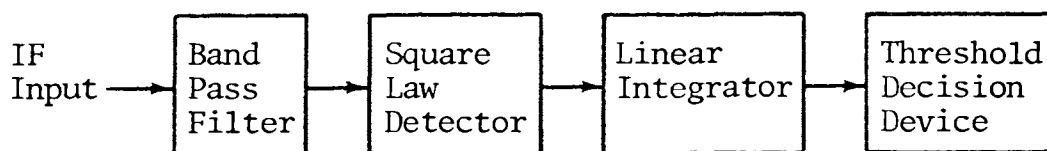


FIGURE 1

Block diagram of receiver

The noise voltage at the input to the linear narrow band filter can be written only as a PDF, because it is statistical in nature. It is given by (6).

$$f(V) = (2\pi\sigma^2)^{-1/2} \exp(-V^2/2\sigma^2) \quad (6)$$

V is the amplitude of the noise voltage at the filter input, and σ^2 is the average input noise power per cycle, $k T_r \bar{N}F$. If a sine wave of amplitude P whose frequency lies in the passband of the filter is added to the noise voltage, the envelope of the output of the narrow-band filter (as derived in Appendix A) is given by the PDF in (7).

$$f(R) = (R/\psi_0) \exp[-(R^2 + P^2)/2\psi_0] I_0(RP/\psi_0) \quad (7)$$

ψ_0 is the average noise power at the filter output, and is equal to $k T_r \bar{N}F B$, as previously stated. R is the amplitude the noise envelope would have at the filter output if no signal were present, and I_0 has been previously defined. Letting $v = R/\sqrt{\psi_0}$ and $a = P/\sqrt{\psi_0}$, (7) becomes

$$f(v) = v \exp[-(v^2 + a^2)/2] I_0(va) \quad (8)$$

If the square law detector follows (9), (8) may be written as (10).

$$e_{out} = e_{in}^2/2 \quad (9)$$

$$f(y) = \exp[-(y+x)] I_0[2(xy)^{1/2}] \quad (10)$$

$y = v^2/2$, $x = a^2/2$, and both are outputs of the square law detector. x is also equal to $P^2/2\psi_0$, or the signal-to-noise power ratio (average noise power at the filter output). Note that (10) reduces to $\exp(-y)$ when no signal is present.

The distribution of N signal-plus-noise or N noise only pulses

may be found through use of the characteristic function. This is defined as $C(p) = E[\exp(py)]$ for the variable y . $E(\)$ is the expectation function. Observe that $C(p)$ is merely the two-sided Laplace transform of the function $f(y)$, with $-s$ replaced by p . i.e.,

$$C(p) = \int_{-\infty}^{\infty} f(y) \exp(py) dy \quad (11)$$

The characteristic function of (10), the distribution of the output of the square law detector, can be obtained from pair 655.1 of Campbell and Foster.⁽¹³⁾ They use $\exp(-py)$ for their first integration, and this notation will be adhered to throughout this dissertation, for it transforms the characteristic function into the standard two-sided Laplace transform with which most engineers are familiar. The characteristic function is given by

$$C(p) = \exp [-xp/(p+1)] / (p+1) \quad (12)$$

This reduces to $1/(p+1)$ when noise only is present. It is well known that the characteristic function of the sum of N independent random variables is obtained by taking the product of the characteristic function of each of the random variables. Assuming the integrator gives equal weight to each of the pulses, the characteristic function of the sum of N pulses is given by (13). This assumes that the noise voltage at the input to the filter is decorrelated in a length of time at least as short as the time between returns, which is true when the reciprocal of the noise bandwidth is greater than the pulse repetition rate.

$$C_N(p) = \left\{ \exp[-xp/(p+1)] / (p+1) \right\}^N \quad (13)$$

This equation neglects the effect of beam shape during a radar scan period. However, it may be taken into account by proper weighting of the pulses at the integrator if desired. It most certainly complicates the analysis, and will not be considered here since it can be handled by including an additional loss factor in (5).

The PDF of the sum of N pulses may now be obtained by finding the inverse transform of (13). This is also found in Campbell and Foster, pair 650.0. The result is given in (14).

$$f_{s,n}(z) = (z/Nx)^{(N-1)/2} [\exp(-z-Nx)] I_{N-1}[2(Nxz)^{1/2}] \quad (14)$$

$I_{N-1}()$ is the modified Bessel function of first kind and order $N-1$. z is the linear sum of N pulses from the square law detector. The density function for noise alone may be found from the inverse transform of (13), with $x = 0$. The result is

$$f_n(z) = z^{N-1} \exp(-z)/(N-1)! \quad (15)$$

This may be obtained from pair 431 of Campbell and Foster, or most any other table of Laplace transforms.

The probability of false alarm (α or the probability of a Type I error in statistical terms) is found by integrating the PDF given by (15) from some threshold, Y_b , to infinity. This is the probability that the sum of N noise variates alone will exceed the threshold. In actual practice, P_f (the probability of false alarm) is usually fixed and Y_b calculated from (16).

$$P_f = \int_{Y_b}^{\infty} z^{N-1} \frac{\exp(-z)}{(N-1)!} dz \quad (16)$$

The probability of detection (P_d) is now found from (17). P_d is also known as one minus the probability of a Type II error or $1-\beta$ in statistical terms.

$$P_d = \int_{Y_b}^{\infty} (z/Nx)^{(N-1)/2} [\exp(-z-Nx)] I_{N-1} [2(Nxz)^{1/2}] dz \quad (17)$$

Marcum applied approximations to obtain P_f and P_d which will not be given here. With the advent of high speed computers it is no longer necessary to utilize approximations when closed form solutions exist which can be easily computed, as they do for (16) and (17).

(3)

The next major published advance was Swerling's work in 1956. Unlike Marcum, he considered targets whose voltage returns were fluctuating. Specifically, he considered four cases, which are listed below.

CASE I

The signal power reflected from the target is constant for the time on target during any one scan, but fluctuates independently from scan-to-scan. In statistical terms, the correlation function relating the amplitudes of the reflected signal pulses to one another is one for the time in which the beam is on target during any one scan, but is zero for a time as long as the interval between scans. The PDF for the input signal to average noise power ratio may be written as (18), which is just (1) repeated for clarity.

$$f(x | \bar{x}) = \exp(-x/\bar{x})/\bar{x} \quad x \geq 0 \quad (18)$$

x = input signal to filter output noise power ratio.

\bar{x} = average of x over all target fluctuations.

CASE II

The PDF of the target return fluctuations is given by (18). However, the returns are independent from pulse-to-pulse, not scan-to-scan as in Case I.

CASE III

The returns are independent from scan-to-scan, but completely correlated during any one scan time, as in Case I. The PDF of the returns is given by (19), where all terms have been previously defined.

$$f(x | \bar{x}) = 4x \exp(-2x/\bar{x}) / \bar{x}^2 \quad x \geq 0 \quad (19)$$

CASE IV

The PDF of the target returns is given by (19), but the returns are independent from pulse-to-pulse, as in Case II.

Cases I and II are most often utilized to model the returns of complex targets such as jet and propeller driven aircraft, respectively. Cases III and IV are more useful for modeling the returns of objects that are approximately spherical, at fairly large wavelengths. Case I and III are also known as scan-to-scan independent, while II and IV are commonly denoted pulse-to-pulse independent.

This dissertation is concerned with targets of the Class I and II variety, with additional modifications which are more in agreement with experimental observations.^(7,8) For this reason, equations for P_d will be developed for these two cases. The equation for P_f , (16), remains the same throughout this dissertation and will receive no more attention, except in the calculation of Y_b .

The characteristic function for the envelope of the square law detector output in the receiver under consideration, for one pulse, was given by (12). Note that this function applies only in the case of constant signal-to-noise ratio, x . For a target with fluctuating returns which can only be described by a PDF, the characteristic function must be modified by averaging over all possible values of x . Therefore, the characteristic function for the PDF of the envelope of the square law detector output is obtained from (20), which is (12) multiplied by (18) and averaged over all possible values of the variable x .

$$\overline{C(p)} = \int_{-\infty}^{\infty} \left\{ \exp[-xp/(p+1)] / (p+1) \right\} \left[\exp(-x/\bar{x}) / \bar{x} \right] dx \quad (20)$$

x ≥ 0

After performing the integration, (21) results.

$$\overline{C(p)} = 1/[1 + p(1+\bar{x})] \quad (21)$$

Now if all returns are pulse-to-pulse independent (Swerling's Case II), the characteristic function of the sum of N signal plus noise variates is given by (22).

$$\overline{C_N(p)} = [p(\bar{x}+1)+1]^{-N} \quad (22)$$

The PDF of the linear integrator output is again found from pair 430 of Foster and Campbell or most any other table of Laplace transforms. The result is

$$f_{s,n}(z) = \frac{z^{N-1} \exp[-z/(1 + \bar{x})]}{(1 + \bar{x})^N (N-1)!} \quad (23)$$

This result applies only to returns that are pulse-to-pulse independent, Swerling's Case II model.

The PDF of the linear integrator output for Swerling's Case I is derived in a different manner. The starting place is the equation of the characteristic function for the sum of N signal plus noise variates, which is given by (13). Multiplying (13) by (18) and averaging over all possible values of x yields (24), the characteristic function desired for this case.

$$\begin{aligned}\overline{C_N(p)} &= \int_{-\infty}^{\infty} [\exp(-x/\bar{x})] \left\{ \exp[-xp/(p+1)]/(p+1) \right\}^N dx \\ &= (1+p)^{1-N}/[1+p(1+N\bar{x})]\end{aligned}\tag{24}$$

Pair 581.7 of Campbell and Foster gives the inverse transform of (24), which is the PDF of the linear integrator output. However, it will be derived in a form more suitable for programming later on, and its presentation will be deferred until then.

Curves of P_d for $N=10$ and 100 , Swerling's Case I and II, and $P_f = 10^{-6}$, are presented in Figure 2 versus the normalized range parameter R/R_0 . Note that P_d is greater for pulse-to-pulse independence (Case II) for $N=10$ than it is for $N=100$ and scan-to-scan independence (Case I), when P_d is greater than 90%. Also, there is a large difference between the two cases in required average signal-to-noise ratio at $P_d = 90\%$, even for N as small as 10 (≈ 7.5 db). Obviously, less transmitter power is required for a given P_d (P_d greater than 40%) when the returns are pulse-to-pulse independent and N is fixed. One technique for accomplishing pulse-to-pulse decorrelation is frequency agility. This is discussed in the next chapter.

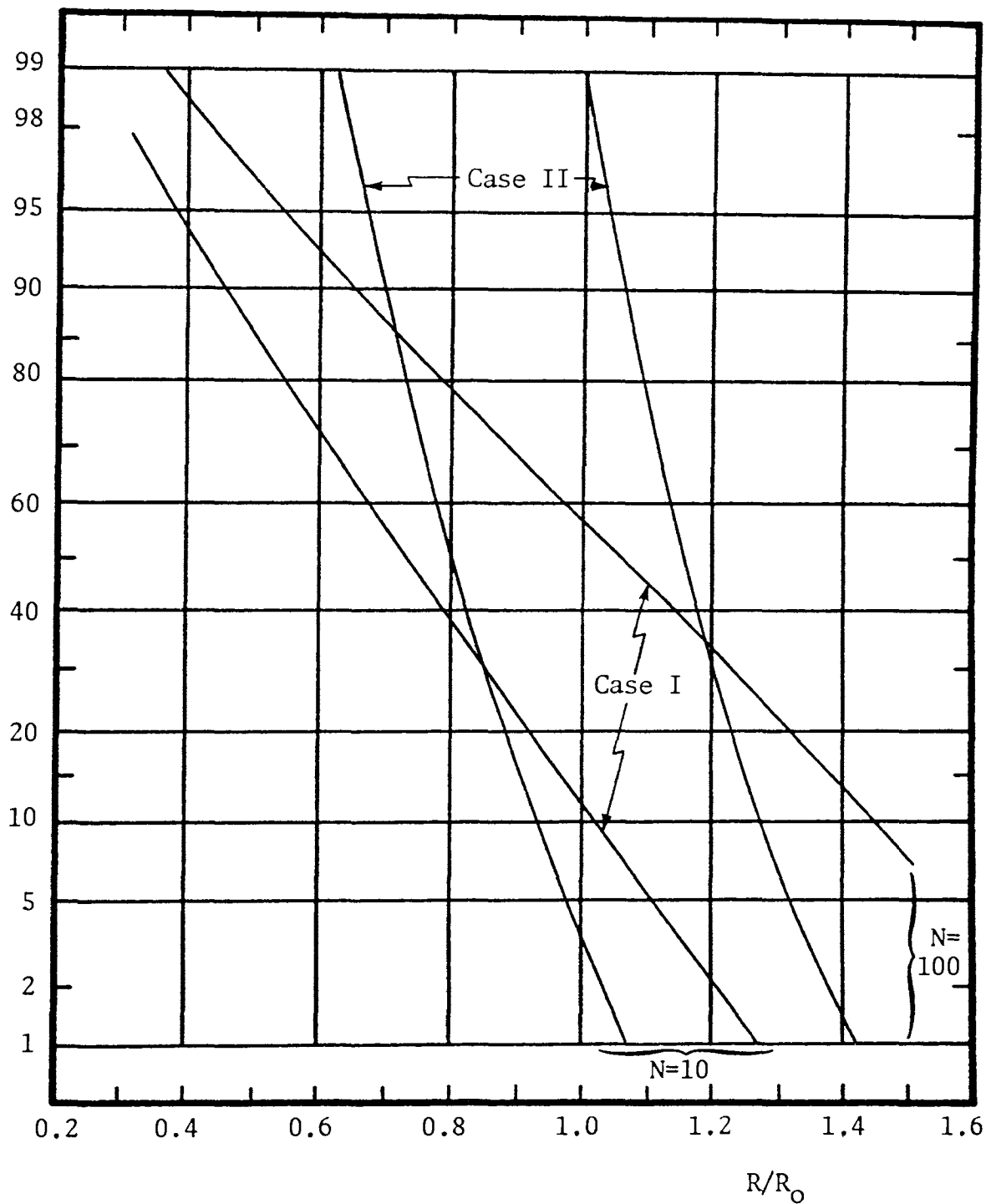
P_d in %

FIGURE 2

Probability of detection vs. R/R_0 for
 $N=10$ and 100 , Swerling's Case I and II,
 probability of false alarm = 10^{-6}

CHAPTER II

SEARCH RADAR WITH FREQUENCY AGILITY

A. Completely Decorrelated Returns

Figure 2 gives an indication of the improvement that can be expected in P_d when the returns are pulse-to-pulse independent (Swerling's Case II) as opposed to scan-to-scan independence (Case I). If a method were available to decrease the correlation between pulses, P_d could be expected to increase for a given transmitter power level. In an effort to decrease the correlation between target returns, it was proposed to utilize a frequency agile transmitter.⁽¹⁾ The correlation between two voltage returns whose frequency is f_i and f_j is given by (25), for a distributed target whose radial depth is D .

$$R(f_i - f_j; D) = \frac{\sin[2\pi D(f_i - f_j)/c]}{2\pi D(f_i - f_j)/c} \quad (25)$$

This equation is derived in Appendix B. The dimension D is illustrated in Figure 3. c is the velocity of light, 3×10^8 meters/second.

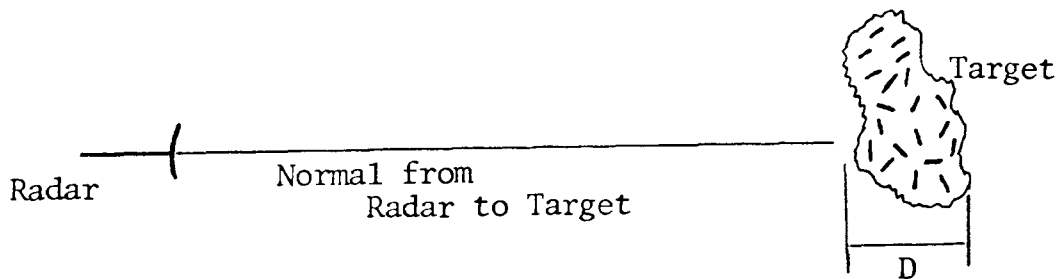


FIGURE 3

Illustration of target depth, D

Note that the first null of (25) occurs when

$$2\pi D(f_i - f_j)/c = \pm\pi \quad (26)$$

The target returns may be considered completely decorrelated when the frequency shift between the transmitted pulses is at least as large as that indicated in (26). The minimum required frequency shift is $150/D$ megahertz per second, where D is the radial target depth in meters. Note that the frequency shift between pulses must be a monotonic function of time in order to prevent any of the returns from becoming correlated during the scan time. This implies that a transmitter bandwidth of 300 megahertz is required in order to convert a Case I target to Case II, when the target depth is 5 meters and 10 pulses are integrated. P_d may be read directly from Swerling's Case II curves for this example. This is the frequency agility technique previously discussed in the literature. This dissertation presents a frequency agility technique differing from the above, which requires a transmitter with a bandwidth of less than 50 megahertz to achieve almost the same values of P_d , for the same example.

High power transmitters with wide bandwidths are more difficult to construct and less efficient than narrow band transmitters. What happens to P_d when the available transmitter bandwidth is insufficient to achieve complete decorrelation of the returns? To answer this question, a technique for calculating P_d for returns that are partially correlated must be formulated. Swerling presented a technique to find the characteristic function for the sum of N partially correlated returns in 1957.⁽¹⁴⁾ This technique will be described in the next section.

B. Partially Correlated Returns

The characteristic function of the linear integrator output is derived for partially correlated returns in this section. Most of this derivation is taken from Swerling's article⁽¹⁴⁾, with intermediate steps supplied by the author of this work. x has already been defined as the ratio of the input signal power per pulse to average noise power, ψ_o , out of the narrow band filter of the receiver under consideration. Now define $u_{k,i}$ as

$$u_{k,i} = V_{k,i}(f_i)/\sqrt{2\psi_o} \quad (27)$$

$V_{k,i}(f_i)$ is defined in Appendix B as the voltage return of the i th pulse at frequency f_i , for a distributed target, and is shown to be a normally distributed random variable with mean zero. This implies $u_{k,i}$ is a normally distributed random variable with mean zero and variance σ_u^2 . It is assumed that the variance of $u_{k,i}$ is the same for all k,i .

Now assume that x_i is of the following form, where i denotes the i th pulse.

$$x_i = u_{1,i}^2 + u_{2,i}^2 \quad (i = 1, 2, \dots, N) \quad (28)$$

This implies that P_i , the amplitude of the sine wave for the i th pulse return as defined in (7), is given by $(V_{1,i}^2 + V_{2,i}^2)^{1/2}$. Also, assume that the $u_{k,i}$ are independent in the subscript i . This implies that the components of x_i (which are $u_{1,i}$ and $u_{2,i}$) are independent during any pulse but $u_{1,i}$ and $u_{1,j}$ are not independent. This will yield partially correlated returns. i.e., the x_i will be partially correlated. Note that this does not affect the analysis of Appendix

B, for the correlation of the voltage returns was found there, not the correlation between signal power returns. All that is implied by (28) is that the power return is actually the sum of two independent voltage components.

If y_i is defined as x_i/σ_u^2 , then y_i is the sum of two normal independent random variables with mean zero and variance one, which are squared. Under these conditions, y_i is distributed as a chi-square random variable with two degrees of freedom. By repeating the substitution $x_i = y_i \sigma_u^2$, the PDF of x_i is found to be given by (29).

$$f(x_i) = \exp(-x_i/2\sigma_u^2)/(2\sigma_u^2) \quad x_i \geq 0 \quad (29)$$

This is known as the exponential distribution and can be written as

$$f(x_i | \bar{x}_i) = \exp(-x_i/\bar{x}_i)/\bar{x}_i \quad x_i \geq 0 \quad (30)$$

because $2\sigma_u^2$ is the expected value of x_i . Note that this is (18), or the PDF utilized to describe the target fluctuations in Swerling's Case I and II target.

Now define the random vector $U^{(k)}$ as

$$U^{(k)} = (u_{k,1}, u_{k,2}, \dots, u_{k,N}) \quad (k=1,2) \quad (31)$$

and assume $U^{(1)}$ is independent of $U^{(2)}$. This is a result of the assumption that $u_{1,i}$ and $u_{2,i}$ are independent. Further, the covariance matrix of $U^{(k)}$ has elements

$$\phi_{i,j}^{(k)} = E(u_{k,i} u_{k,j}) \quad \begin{matrix} k = 1,2 \\ i,j = 1,2,\dots,N \end{matrix} \quad (32)$$

The covariance matrix of $U^{(k)}$ will be denoted $\phi^{(k)}$ and it will be

assumed that $\phi(1) = \phi(2) = \phi$. This assumption is valid as long as the target does not undergo any rapid changes in orientation, which is not expected when the pulse repetition rate of the radar is on the order of 200 hertz per second or greater.

The characteristic function of the output of the linear integrator, for known signal returns, is found from (12) in the same manner as (13) was derived. Here the x_i are not constant, and the characteristic function is given by (33).

$$C(p|x_1, x_2, \dots, x_N) = \prod_{i=1}^N \exp[-px_i/(1+p)]/(p+1) \quad (33)$$

Replacing the x_i with its equivalent from (28), (33) becomes

$$C(p|x_1, x_2, \dots, x_N) = \prod_{i=1}^N \exp[-p(u_{1,i}^2 + u_{2,i}^2)/(p+1)]/(p+1) \quad (34)$$

$C(p)$ is found by averaging (34) over the PDF of $U^{(k)}$. Recalling that $U^{(k)}$ is a normal independent random vector with mean zero and covariance matrix ϕ , the PDF of $U^{(k)}$ may be written as

$$f[U^{(k)}] = (2\pi)^{-N/2} \Delta_k^{-1/2} \exp[-U^{(k)} \Omega (U^{(k)})^T / 2] \quad (35)$$

where Ω is the inverse of the ϕ matrix, the superscript T implies the transpose of the matrix, and Δ_k is the determinant of the covariance matrix, ϕ . Recalling that $\phi(1) = \phi(2) = \phi$, and expanding the product, $C(p)$ may be written as (36).

$$\begin{aligned}
C(p) = & (p+1)^{-N} \int_{U(1)} \int_{U(2)} \exp \left[\frac{-p}{p+1} (u_{1,1}^2 + u_{2,1}^2 + \dots + u_{1,N}^2 + u_{2,N}^2) \right] \\
& \frac{1}{(2\pi)^N \Delta} \exp \left\{ \frac{-1}{2} [U^{(1)} \Omega (U^{(1)})^T + U^{(2)} \Omega (U^{(2)})^T] \right\} dU^{(1)} dU^{(2)}
\end{aligned} \tag{36}$$

Combining the exponential terms, (36) reduces to (37), where I is the identity matrix.

$$\begin{aligned}
C(p) = & (p+1)^{-N} \frac{1}{(2\pi)^N \Delta} \int_{U(1)} \int_{U(2)} \\
& \exp \left\{ \frac{-1}{2} [U^{(1)} \left(\Omega + \frac{2p}{p+1} I \right) U^{(1)T} + U^{(2)} \left(\Omega + \frac{2p}{p+1} I \right) U^{(2)T}] \right\} \\
& dU^{(1)} dU^{(2)}
\end{aligned} \tag{37}$$

If the numerator and denominator of (37) is multiplied by the determinant of the inverse of $[\Omega + 2pI/(p+1)]$, which is denoted as $\Delta(p)$, $C(p)$ reduces to (38) because the integral is immediately recognized as the integral over all space of the joint PDF of two Gaussian random variables and integrates to one.

$$C(p) = (p+1)^{-N} \Delta(p) / \Delta \tag{38}$$

(38) is solved most easily by inverting the determinants, in order to have both as functions of Ω .

$$\frac{\Delta(p)}{\Delta} = \frac{\text{Det}(\Delta^{-1})}{\text{Det}[\Delta(p)^{-1}]} = \frac{\text{Det}(\Omega)}{\text{Det}[\Omega + 2pI/(p+1)]} \tag{39}$$

$\text{Det}(\)$ denotes the determinant of the function in the brackets. it is well known that the value of a positive definite symmetric determinant (which the numerator and denominator of (39) are) is given by the product of the eigenvalues of the determinant. Further, it is immediately obvious that if λ_i are the eigenvalues of Ω , that $\lambda_i + 2p/(p+1)$ are the eigenvalues of the matrix $[\Omega + 2pI/(p+1)]$. Denoting the eigenvalues of Ω by λ_i , $C(p)$ may be written as

$$C(p) = \prod_{i=1}^N (p+1)^{-1} \frac{\lambda_i}{\lambda_i + 2p/(p+1)} = \prod_{i=1}^N (p+1+2p/\lambda_i)^{-1} \quad (40)$$

It is easily shown that the eigenvalues of Ω , λ_i , are the reciprocal of the eigenvalues of Φ , which will be denoted as μ_i

$$\begin{aligned} [A - \lambda I] X &= 0 & AX &= \lambda X \\ X &= A^{-1} \lambda X & [A^{-1} - \frac{I}{\lambda}] X &= 0 \end{aligned} \quad (41)$$

$C(p)$ may now be written in the form

$$C(p) = \prod_{i=1}^N (p+1+2p\mu_i)^{-1} \quad (42)$$

where the μ_i are the eigenvalues of Φ , the covariance matrix of $U^{(k)}$. Now if the eigenvalues of the normalized covariance matrix, Φ/σ_u^2 , are defined as μ_i , $C(p)$ becomes

$$C(p) = \prod_{i=1}^N (p+1+2\sigma_u^2 \mu_i p)^{-1} \quad (43)$$

Recalling that the expected value of x_i , \bar{x}_i , is given by $2\sigma_u^2$, (43) may be written as (44).

$$C(p) = \prod_{i=1}^N (p+1+p \bar{x}_i \mu_i)^{-1} \quad (44)$$

The subscript i may be dropped by assuming $\bar{x}_i = \bar{x}$, for all i . This assumption results from neglecting the beamshape of the antenna. The normalized covariance matrix, Φ/σ_u^2 , is known as the correlation matrix. Note that with the definition of (27), the terms of the correlation matrix may be found through the correlation function derived in Appendix B, which is given by (25). $C(p)$ becomes

$$C(p) = \prod_{i=1}^N (p+1+p \bar{x} \mu_i)^{-1} \quad (45)$$

where μ_i is the i th eigenvalue of the correlation matrix. This equation for the characteristic function will be utilized in the remainder of this dissertation. Note that (45) reduces to (24) when the returns fluctuate from scan-to-scan only (Swerling's Case I), for the eigenvalues of a matrix of one's are given by $\mu_1 = N$, $\mu_2 = \mu_3 = \dots = \mu_N = 0$. (45) also reduces to (22) when the returns are pulse-to-pulse independent (Swerling's Case II) because all of the eigenvalues are equal to one for an identity matrix.

In the next chapter, the off-diagonal components of the correlation matrix will be found with the aid of (25). (The diagonal components will always be equal to unity.) These components will depend on the type of transmitter employed in the radar. Two practical transmitters will be discussed, and solutions for both will be obtained.

CHAPTER III

CALCULATION OF THE CORRELATION COEFFICIENT

A. Description of Practical Transmitters

This chapter is devoted to calculating the correlation coefficients between the voltage returns for two types of practical transmitters. As previously stated, the correlation coefficients are the off-diagonal terms of the correlation matrix, or normalized covariance matrix. Once these terms are found, they will be utilized to find the eigenvalues of the correlation matrix. The eigenvalues will then be substituted into (45) to find the characteristic function of the linear integrator output.

There are several types of transmitters which may be utilized in frequency agile radar systems. Two of the most common are the dither tuned and spin tuned magnetron. The output frequency of the dither tuned magnetron is controlled by an oscillating diaphragm, much like a conventional radio speaker. Because of this tuning method, almost any continuous distribution of frequencies may be obtained from the dither tuned magnetron. The output frequency of the spin tuned magnetron is approximately sinusoidal with time. In general, the spin tuned magnetron has a larger range of possible frequencies, or transmitter bandwidth, than the dither tuned magnetron. Other methods of obtaining frequency shifts between transmitter pulses include modulating the helix of a backward wave oscillator (BWO) with a voltage and selection of fundamental oscillators which are multiplied up to the proper microwave frequency before transmission. The microwave frequency output of both of these devices (BWO and multiplier chain) is usually amplified by a

high power transmitter tube, such as a traveling wave tube, crossed field amplifier, or klystron amplifier, before reaching the transmitting antenna.

The frequencies radiated by the transmitter can only be described by a PDF in this dissertation. It will be shown that random selection of the frequency of each transmitted pulse from a PDF is more effective for target detection than known predetermined frequency shifts between pulses, for a transmitter with constant bandwidth. Two types of frequency distributions will be discussed. The first is a uniform distribution. This is the most effective PDF from an anti-jamming or ECCM viewpoint, for the output frequency of each pulse is equally likely to be any frequency within its possible range. This distribution may be obtained exactly from a modulated BWO or multiplier chain, or approximated by the dither tuned magnetron. The other PDF which will be considered is known as the anti-sine distribution. This is obtained by randomly sampling (in time) a function that is sinusoidal with time. It is also obtained when the starting phase of the sinusoidal function is unknown. Under this condition, the sampling may be made randomly or periodically, provided the period of the sampling time is not equal to the periodicity of the sinusoidal function. This difference in periodicity will be assumed because in a practical application there will always be slippage between the two functions. This slippage results from frequency modulation on the transmitter due to system noise which may arise, for example, from the motor which drives the spin tuned magnetron or the ripple of the BWO power supply.

One other point should be emphasized about these distributions. The time between transmitter pulses should be sufficient to allow the

tuning mechanism, which is employed to shift frequencies between pulses, to travel almost from one extreme to the other, in order to make the assumption that the frequencies are independent between pulses. For example, if a backward wave oscillator was linearly tuned over its possible bandwidth of, for instance, one hundred megahertz, in one second, transmitting pulses at a rate of one thousand pulses per second would invalidate the assumption of random frequency selection between pulses.

B. Correlation Coefficient for the Uniform PDF

For the uniform distribution, the PDF of the transmitter frequency is given by (46).

$$f(f_i) = 1/\Delta f \quad f_0 - \Delta f/2 \leq f_i \leq f_0 + \Delta f/2 \quad (46)$$

f_i is the frequency of the i th transmitter pulse, and f_0 is the center frequency of the transmitter which has a bandwidth of Δf . The correlation coefficient is found by averaging $R(f_i - f_j; D)$, which is given by (25), over the PDF of $f_i - f_j$. The PDF of $f_i - f_j$ was derived through the well known change of variable technique, rather than the characteristic function method, because this was the only method by which a solution could be obtained by the author for the anti-sine distribution.

The joint PDF of f_i and f_j may be written as (47), assuming the frequencies of the transmitter are independent random variables obtained by sampling the uniform PDF of (46).

$$f(f_i, f_j) = 1/(\Delta f)^2 \quad f_0 - \Delta f/2 \leq f_i, f_j \leq f_0 + \Delta f/2 \quad (47)$$

By assuming $f_i \geq f_j$, $w = f_i - f_j$, $t = f_j$, the joint PDF of w and t is given by (48).

$$f(w,t) = 2/(\Delta f)^2 \quad \begin{array}{l} 0 \leq w \leq \Delta f \\ f_0 - \Delta f/2 \leq t \leq f_0 + \Delta f - w \end{array} \quad (48)$$

The PDF of w is obtained by integrating (48) over all possible values of t .

$$f(w) = \int_{f_0 - \Delta f/2}^{f_0 + \Delta f/2 - w} 2/(\Delta f)^2 dt = 2(\Delta f - w)/(\Delta f)^2 \quad 0 \leq w \leq \Delta f \quad (49)$$

Note that (49) is also the PDF of w' , when f_j is assumed greater than f_i , $w' = f_j - f_i$, and $t = f_i$, and w is replaced everywhere by w' in (49). The average correlation coefficient, hereinafter defined as ρ , is given by (50).

$$\rho = \int_0^{\Delta f} \frac{2(\Delta f - w)}{(\Delta f)^2} \frac{\sin(2\pi Dw/c)}{2\pi Dw/c} dw \quad (50)$$

This equation is correct regardless of whether f_i is greater than f_j because w may be replaced by w' without affecting the result. After making the substitution $W = 2\pi D \Delta f/c$, the solution of (50) becomes

$$\rho = \frac{2}{W} \left\{ \text{Si}(W) + [\cos(W) - 1]/W \right\} \quad (51)$$

where $\text{Si}(W) = \int_0^W [\sin(x)/x] dx$

After expanding $\text{Si}(W)$ and $\cos(W)$ into their equivalent power series, (51) becomes (52).

$$\rho = 2 \sum_{i=1}^{\infty} (-1)^{i+1} W^{2i-1} / (2i-1)(2i)! \quad (52)$$

Note that when W approaches zero, which implies zero target depth or fixed transmitter frequency, ρ reduces to unity as required. This equation is plotted in Figure 6 as a function of the product of D in meters and Δf in megahertz, along with the average correlation coefficient for the anti-sine distribution, which is derived in the next section.

C. Correlation Coefficient for the Anti-sine PDF

The anti-sine distribution results when independent random samples (random in time) are taken from a function that is sinusoidal with time. This is illustrated in Figure 4.

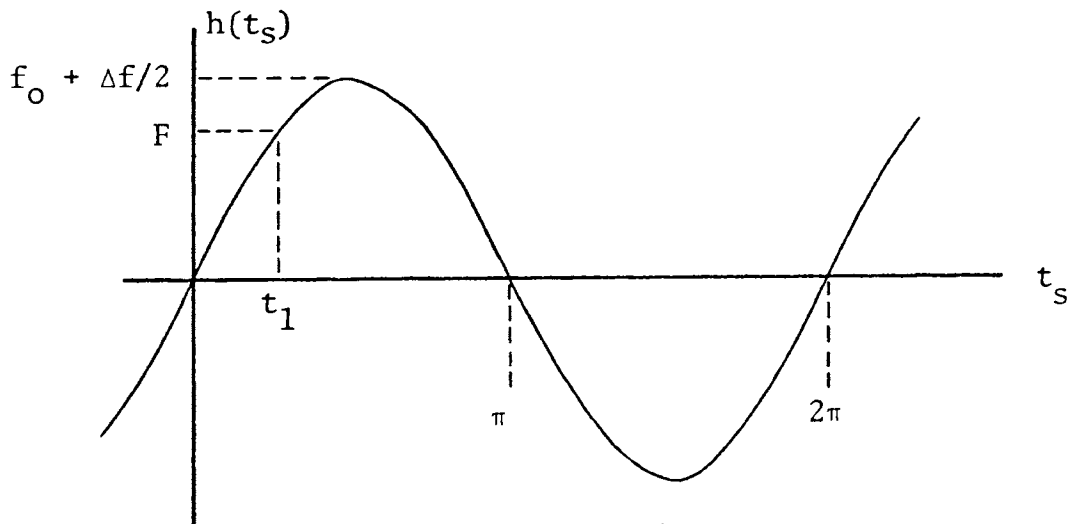


FIGURE 4

Illustration of random sampling from a sine wave

$h(t_s)$ describes the frequency of the transmitter as a function of sampling time, t_s , and is given by (53).

$$h(t_s) = f_0 + [\Delta f \sin(t_s)]/2 \quad (53)$$

This function is representative of the transmitter frequency for a spin tuned magnetron, and can also be obtained from a dither tuned magnetron or a BWO when the helix modulation voltage has the proper form.

The anti-sine distribution is derived from $h(t_s)$ in the following manner. The probability that f_i , the frequency of the i th transmitter pulse, is less than or equal to F , is given by (54).

$$P[f_i \leq F] = 1 - 2 P[t_1 \leq t_s \leq \pi/2] \quad (54)$$

Since t_s , the sampling time, is assumed to be a random variable uniformly distributed between 0 and 2π , (54) may be written as (55).

$$P[f_i \leq F] = 1 - 2 \int_{t_1}^{\pi/2} (1/2\pi) dt_s \quad (55)$$

t_1 is related to the frequency parameters through (56).

$$t_1 = \arcsin[2(F-f_0)/\Delta f] \quad (56)$$

Substituting (56) into (55) and performing the integration yields (57), the cumulative density function of f_i .

$$P[f_i \leq F] = 1 - \left\{ (\pi/2) - \arcsin[2(F-f_0)/\Delta f] \right\} / \pi \quad (57)$$

It is well known that the derivative of the cumulative density function yields the PDF. Therefore, the PDF of f_i is given by (58).

$$f(f_i) = \frac{d}{dF} \left\{ P[f_i \leq F] \right\}_{F=f_i} = \frac{1}{\pi [(\Delta f/2)^2 - (f_i - f_0)^2]^{1/2}} \quad (58)$$

$$f_0 - \Delta f/2 \leq f_i \leq f_0 + \Delta f/2$$

If the sampling is performed periodically instead of randomly, but the starting phase (θ_s in Figure 5) is unknown, the probability that f_i is less than or equal to F is given by (59).

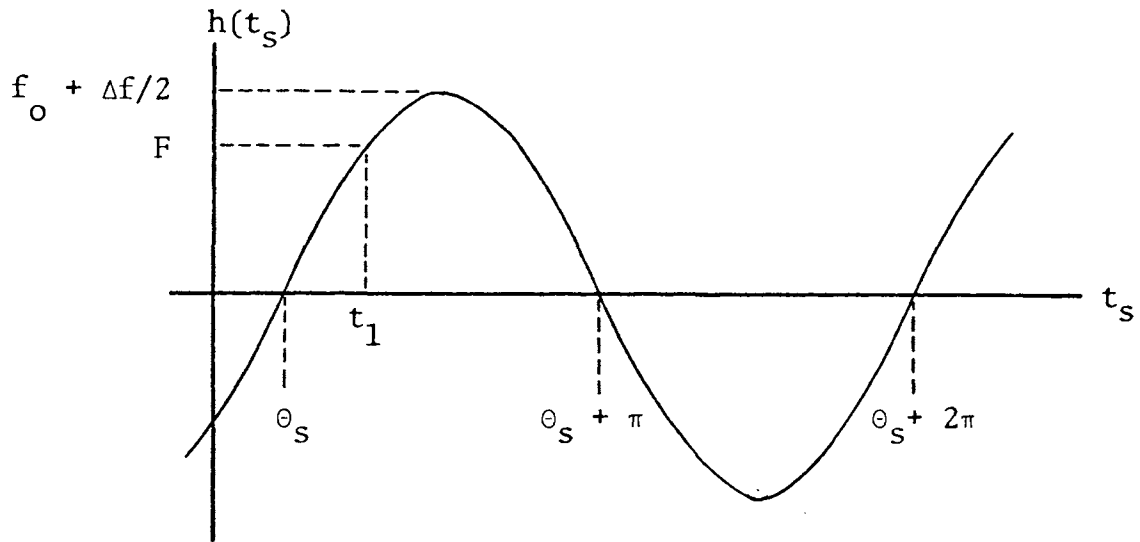


FIGURE 5

Illustration of sampling from a sine wave with unknown starting phase

$$P[f_i \leq F] = 1 - 2 P[t_1 \leq t_s \leq \pi/2 + \theta_s] \quad (59)$$

$h(t_s)$ is given by (60).

$$h(t_s) = f_0 + (\Delta f/2) \sin(t_s - \theta_s) \quad (60)$$

Since t_s is assumed known and θ_s is a random variable assumed to be uniformly distributed from 0 to 2π , $t_s - \theta_s$ is a random variable uniformly distributed between t_s and $t_s + 2\pi$. Under these conditions,

$t_1 = \theta_s + \arcsin[2(F - f_0)/\Delta f]$ and (59) becomes

$$P[f_i \leq F] = 1 - 2 P[t_1 - \theta_s \leq t_s - \theta_s \leq \pi/2] = 1 - 2 \int_{\arcsin[2(F - f_0)/\Delta f]}^{\pi/2} (1/2\pi) d(t_s - \theta_s) \quad (61)$$

After performing the indicated integration, (61) becomes (57) again, as promised. Taking the derivative yields (58), the pdf of f_i . Note that the PDF of f_j is also given by (58) for this case, even though the sampling time is known, and f_j is independent of f_i because of the slippage previously mentioned.

The joint PDF of f_i and f_j is obtained by multiplying their density functions together, because they were assumed independent.

$$f(f_i, f_j) = \frac{1}{\pi^2 [(\Delta f/2)^2 - (f_i - f_o)^2]^{1/2} [(\Delta f/2)^2 - (f_j - f_o)^2]^{1/2}} \quad (62)$$

By defining Q as the absolute value of $f_i - f_j$ and t as the smaller of the two frequencies, the PDF of Q and t becomes

$$f(Q, t) = \frac{2}{\pi^2 [(\Delta f/2)^2 - (Q+t-f_o)^2]^{1/2} [(\Delta f/2)^2 - (t-f_o)^2]^{1/2}} \quad (63)$$

$$0 < Q \leq \Delta f$$

$$f_o - \Delta f/2 \leq t \leq f_o + \Delta f/2 - Q$$

The PDF of Q is found by integrating (63) over all values of t .

$$f(Q) = \int_{f_o - \Delta f/2}^{f_o + \Delta f/2 - Q} f(Q, t) dt \quad 0 < Q \leq \Delta f \quad (64)$$

The integral was evaluated using 3.147.4 of reference (15). The result is

$$f(Q) = (2/\pi)^2 K \{ [1 - (Q/\Delta f)^2]^{1/2} \} / \Delta f \quad 0 < Q \leq \Delta f \quad (65)$$

where $K\{\}$ denotes the complete elliptic integral of the first kind, of order $[1-(Q/\Delta f)^2]^{1/2}$. The series expansion of $K(x)$ was obtained from 17.3.11 of reference (16), and is given in (66).

$$K(x) = \frac{\pi}{2} \left[1 + \left(\frac{1}{2} \right)^2 x^2 + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 x^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 x^6 + \dots \right] \quad (66)$$

$$|x| < 1$$

(66) does not agree exactly with the reference because of the manner in which x is defined. However, (65) and (66) are consistent with each other. The average correlation coefficient was calculated by multiplying (65) by (25), which is $R(f_i - f_j; D)$, and integrating over all values of Q . This is given in (67).

$$\rho = \int_0^{\Delta f} \frac{\sin(2\pi DQ/c)}{2\pi DQ/c} \frac{(2/\pi)^2}{\Delta f} K \left\{ [1-(Q/\Delta f)^2]^{1/2} \right\} dQ \quad (67)$$

No closed form solution was found by the author for (67). Therefore, the series expansion of the complete elliptic integral of the first kind, which is (66), was utilized and the integration was performed numerically on a digital computer utilizing Simpson's rule integration. The result is plotted in Figure 6 along with the average correlation coefficient for the uniform PDF, which was derived in the previous section of this chapter.

Now that the correlation coefficients of the voltage returns have been derived for a transmitter whose frequency can be described only by a PDF, these values may be substituted into the correlation matrix in order to find the eigenvalues of the matrix. Remembering that the average correlation coefficient, ρ , as found in this chapter, applies

Correlation
coefficient,
 ρ

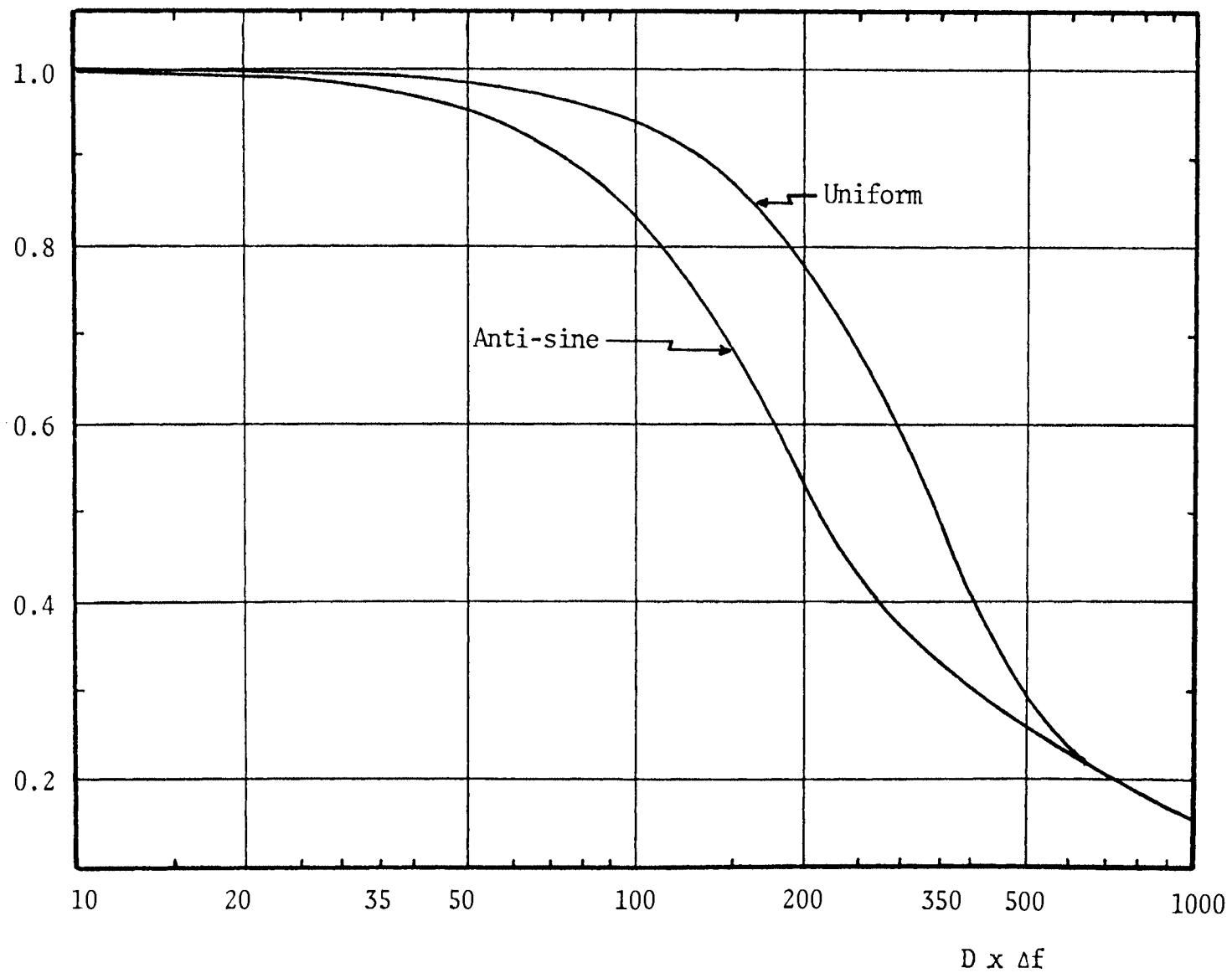


FIGURE 6

Correlation coefficient vs. the product of target depth,
 D , in meters, and transmitter bandwidth, Δf , in megahertz

for any two frequencies, it is obvious that all off-diagonal terms of the correlation matrix are equal to ρ . This fact will be invaluable in calculating the PDF of the integrator output from the characteristic function, (45). The next chapter will be devoted to calculating P_d for the two cases discussed in this chapter.

CHAPTER IV

FREQUENCY AGILE RADAR DETECTION

PROBABILITIES FOR SWERLING'S CASE I TARGET

A. Calculation of the Linear Integrator Output PDF

Swerling's Case I target has previously been designated as scan-to-scan independent, for a fixed frequency radar. The purpose of a frequency agile radar is to create some decorrelation between pulses in order to improve P_d at a given average signal-to-noise power ratio, \bar{x} . If the frequency shift between pulses is sufficiently large, as given by (26), the returns become pulse-to-pulse independent and Swerling's Case II curves may be utilized to find P_d . However, practical transmitters with bandwidth requirements of this magnitude are often unfeasible. In general, efficient transmitters will have a bandwidth which is sufficient for partial decorrelation of the returns.

The correlation coefficient for two types of transmitter frequency distributions and given target depth was derived in the previous chapter. Because of the definition of the correlation coefficient, all of the off-diagonal terms of the correlation matrix are equal to ρ . The characteristic function of the integrator output, for partially correlated returns, is given by (45). In order to find the inverse transform of this function, the eigenvalues of the correlation matrix must be found. In Appendix C, it is shown that the eigenvalues of a matrix whose off-diagonal terms are all equal to ρ , $0 \leq \rho \leq 1$, and whose diagonal terms are all unity, are given by (68).

$$\begin{aligned}\mu_1 &= 1 + (N-1)\rho \\ \mu_2 &= \mu_3 = \dots = \mu_N = 1 - \rho\end{aligned}\quad (68)$$

N is the order of the matrix, and μ_i is the i th eigenvalue. If these eigenvalues are substituted into the equation for the characteristic function, (45), the result is given by

$$\begin{aligned}C(p) &= A B^{N-1} / (A+p)^{N-1} (B+p) \\ A &= \{1 + \bar{x} [1 + (N-1)\rho]\}^{-1} \quad B = [1 + \bar{x}(1-\rho)]^{-1}\end{aligned}\quad (69)$$

Observe that this result holds for every correlation matrix whose off-diagonal terms are all equal to ρ and whose diagonal terms are always unity.

The inverse transform of (69) is most easily found through the partial fraction expansion technique. The partial fraction expansion of (69) is given by (70).

$$C(p) = AB^{N-1} \left\{ [(B-A)^{N-1} (p+A)]^{-1} - \sum_{i=1}^{N-1} (B-A)^{i-N} (p+B)^{-i} \right\} \quad (70)$$

Since the inverse transform of $(B+p)^{-i}$ is given by $z^{i-1} \exp(-Bz)/(i-1)!$, the inverse transform of (70) is given by (71),

$$f(z) = AB^{N-1} \left[\frac{\exp(-Az)}{(B-A)^{N-1}} - \sum_{i=1}^{N-1} \frac{z^{i-1} \exp(-Bz)}{(B-A)^{N-i} (i-1)!} \right] \quad (71)$$

where A and B have already been defined, and z is the output of the linear integrator. This is the PDF of the linear integrator output for partially correlated target returns (Swerling's Case I with a

frequency agile transmitter). This is also the inverse transform of (24) if $\rho=1$, as promised in Chapter I.

B. Calculation of P_d Utilizing (71)

P_d is now found by integrating (71) from some bias level, Y_b , to infinity. Y_b is found from (16), for a fixed P_f . This is easily accomplished on a digital computer, and is included in the program of Appendix D. Integration of (71) is easily accomplished by utilizing the recurrence relation of (72).

$$\int_{Y_b}^{\infty} t^{i-1} \exp(-Bt) dt = Y_b^{i-1} \exp(-BY_b)/B + (i-1)/B \int_{Y_b}^{\infty} t^{i-2} \exp(-Bt) dt \quad (72)$$

P_d is given by (73).

$$P_d = A B^{N-1} \left[\exp(-AY_b)/A(B-A)^{N-1} - \exp(-BY_b) \sum_{i=1}^{N-1} (B-A)^{i-N} Y_b^i (BY_b)^{-j} / (i-j)! \right] \quad (73)$$

Observe that the solution is a finite series and is easily programmed on a digital computer.

Curves of P_d versus R/R_0 for different values of ρ , P_f , and N , are plotted in Figures 7 through 15. Three values of N were chosen, $N=10$, 25, and 100. This range is sufficient to include most radars. Three values of P_f , 10^{-6} , 10^{-8} , and 10^{-10} , were also chosen to be representative of most radars. The equations presented in this section may be utilized in cases where other parameters are of interest. Table I

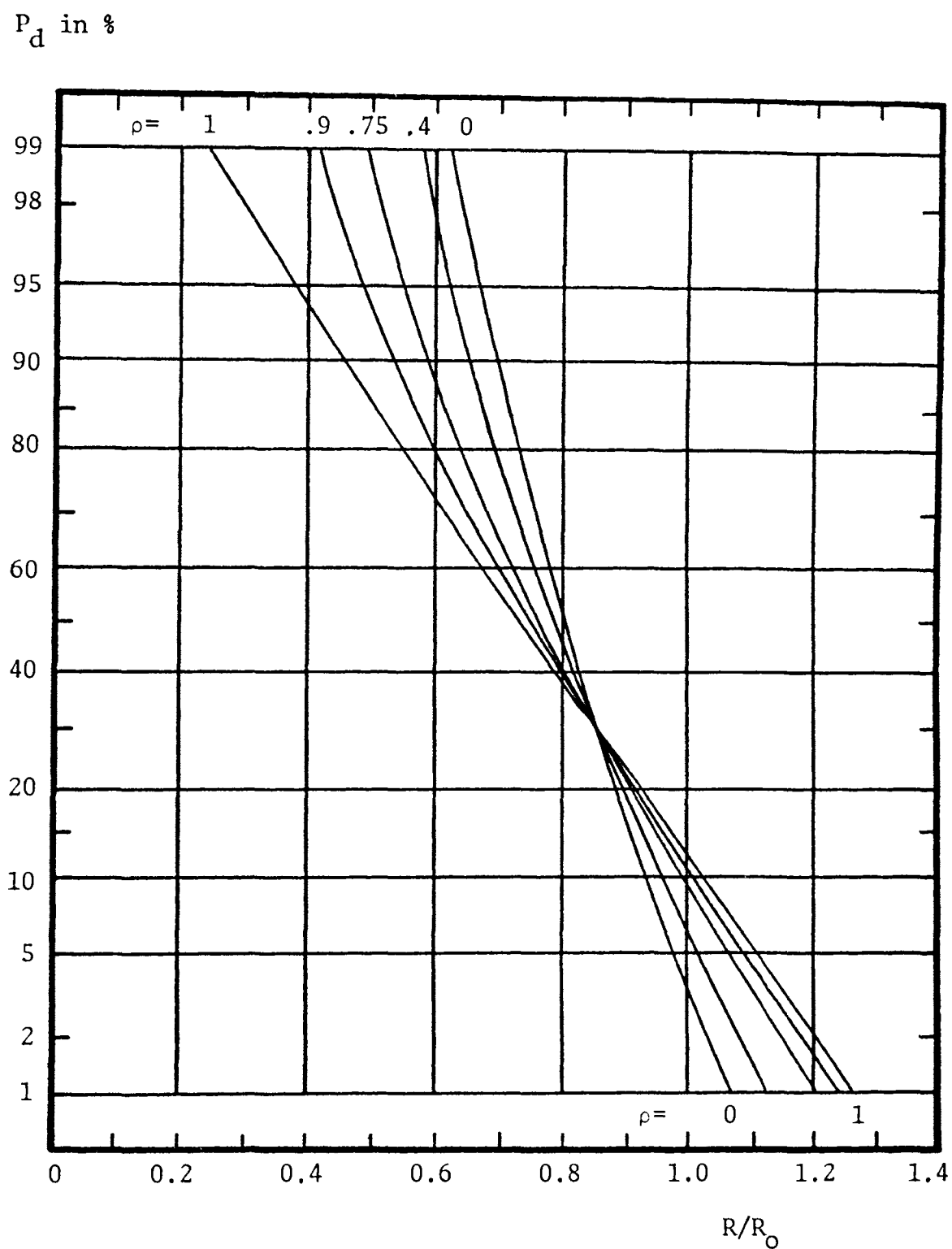


FIGURE 7

Probability of detection vs. R/R_0 for $N = 10$, probability of false alarm = 10^{-6}

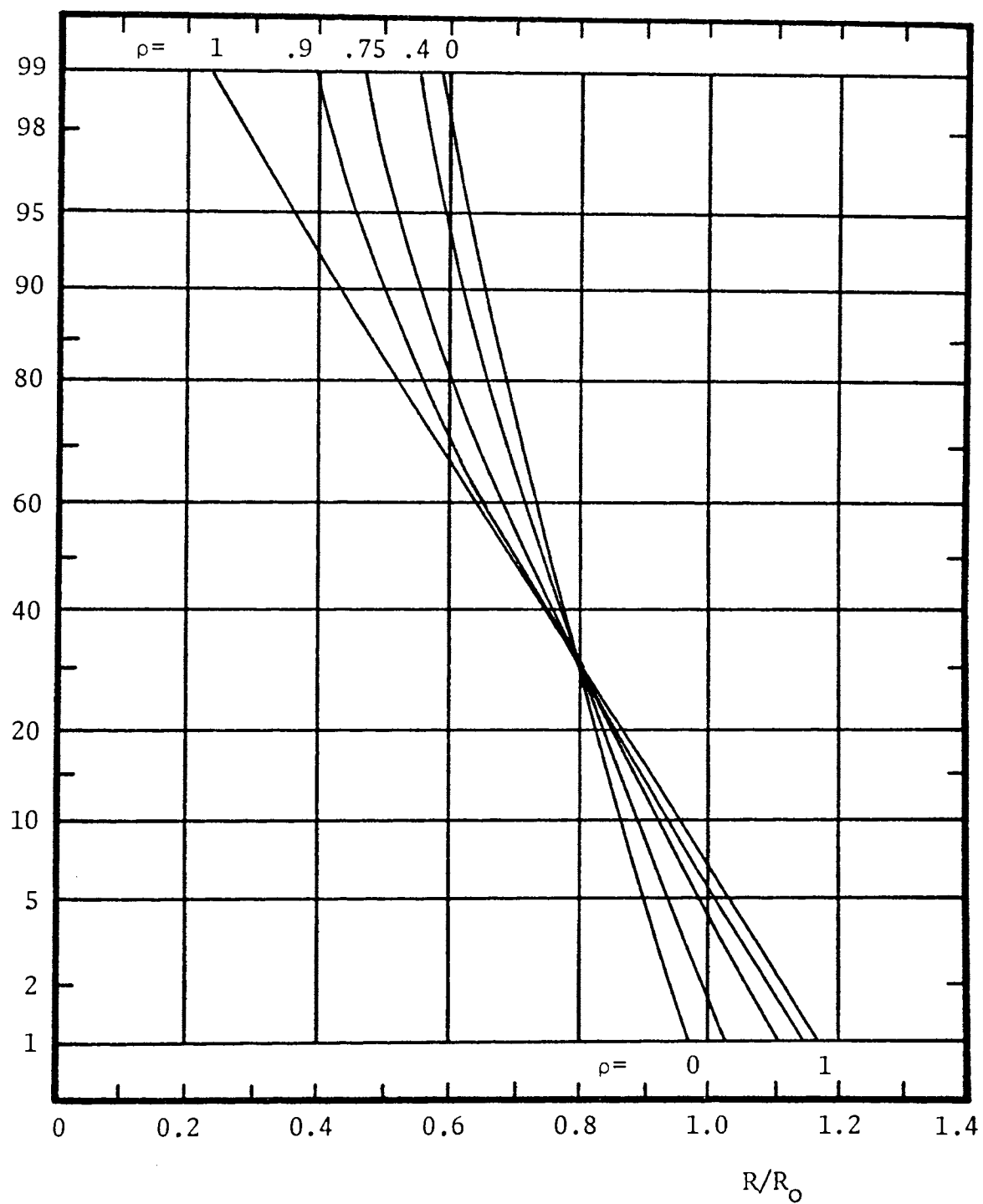
P_d in %


FIGURE 8

Probability of detection vs. R/R_0 for
 $N = 10$, probability of false alarm = 10^{-8}

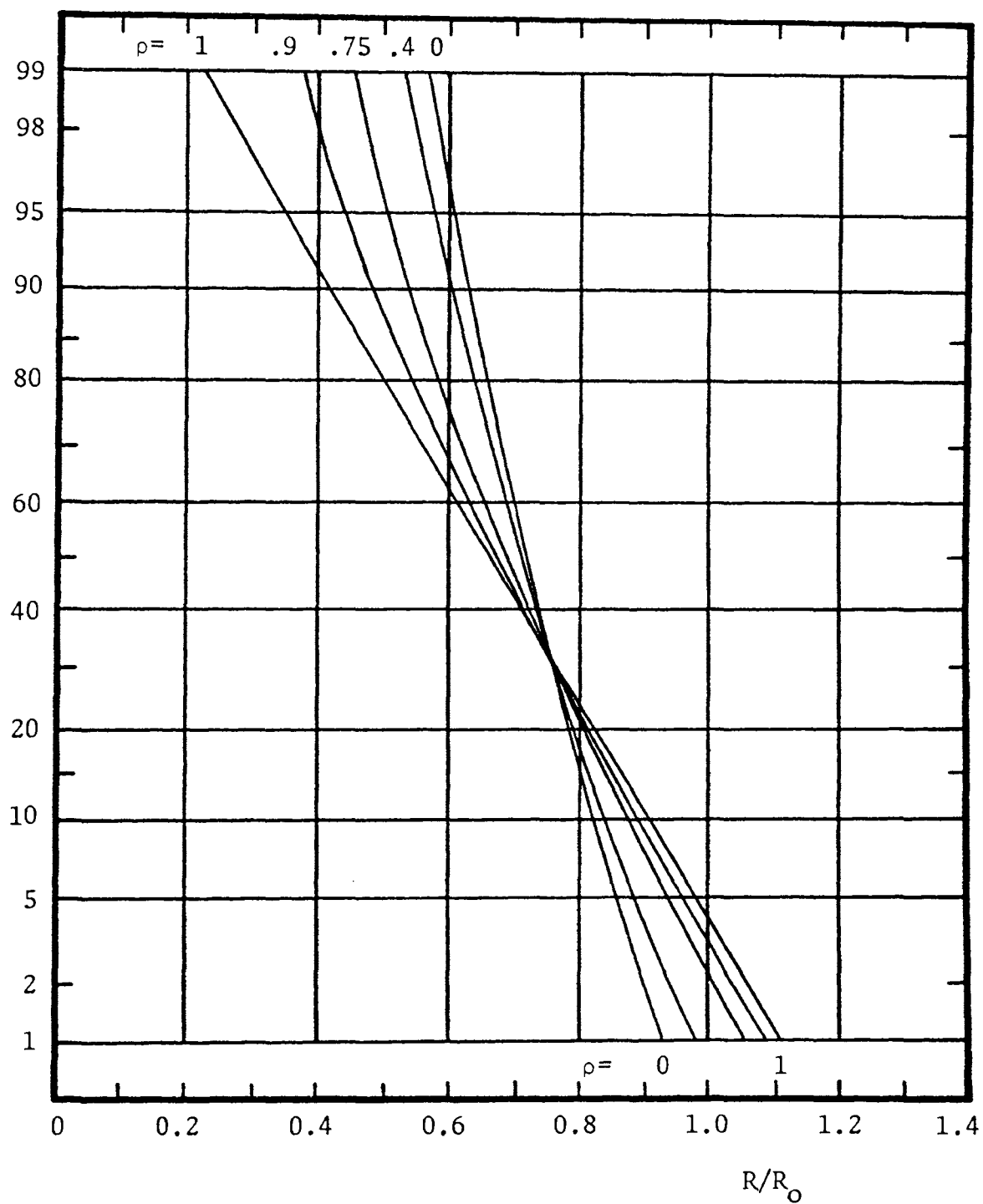
P_d in %


FIGURE 9

Probability of detection vs. R/R_0 for
 $N = 10$, probability of false alarm = 10^{-10}

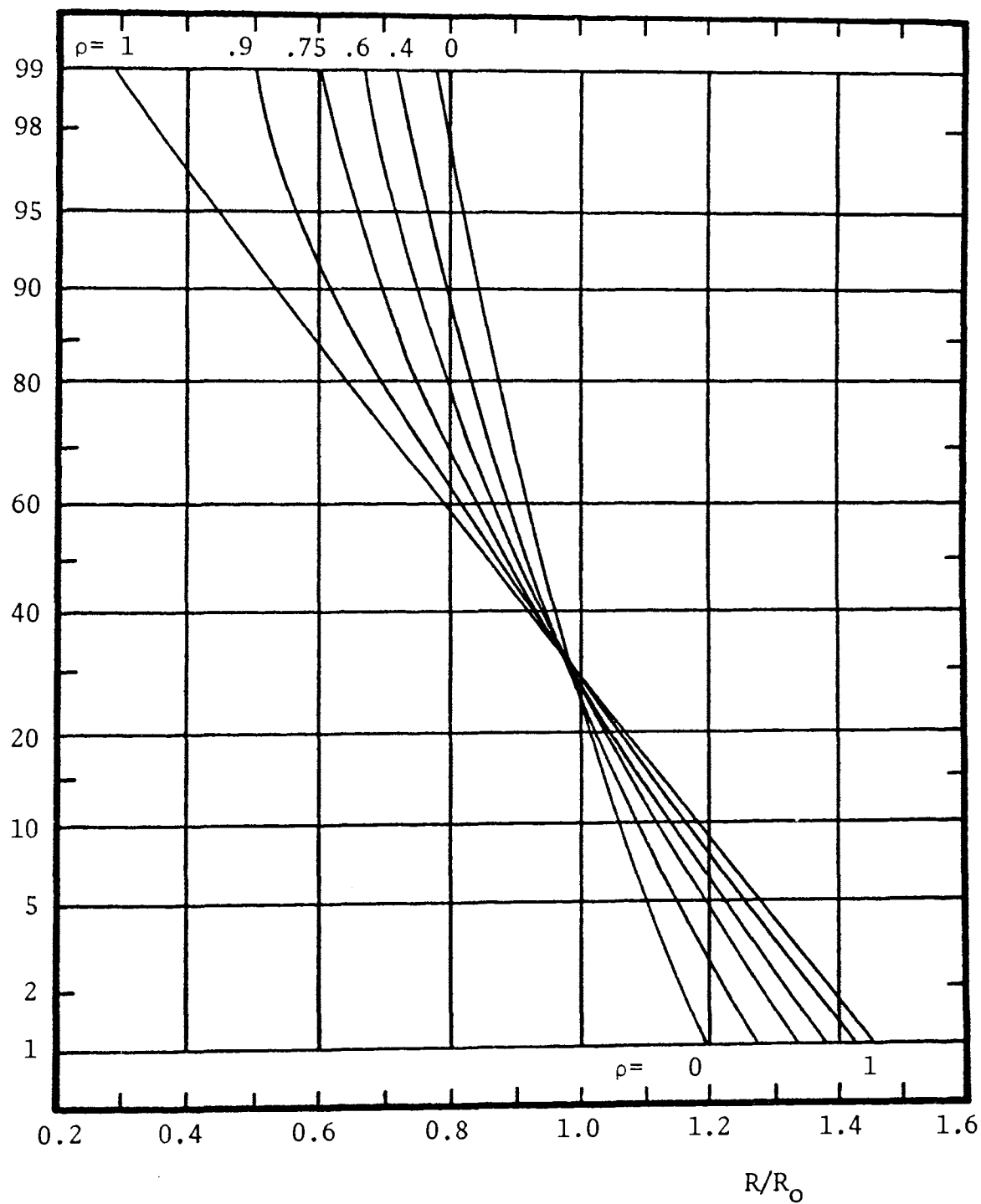
P_d in %


FIGURE 10

Probability of detection vs. R/R_0 for
 $N = 25$, probability of false alarm = 10^{-6}

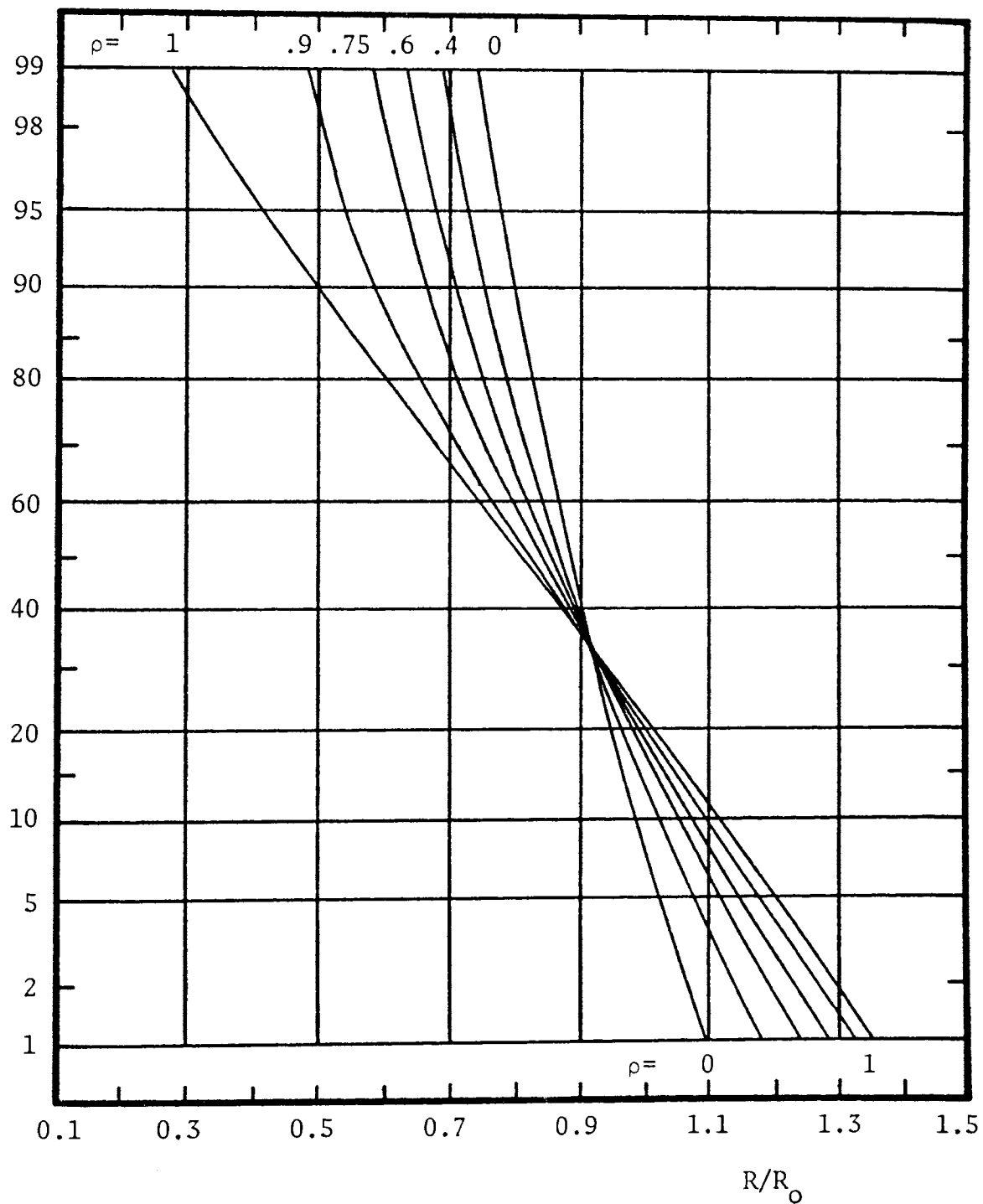
P_d in %


FIGURE 11

Probability of detection vs. R/R_0 for $N = 25$, probability of false alarm = 10^{-8}

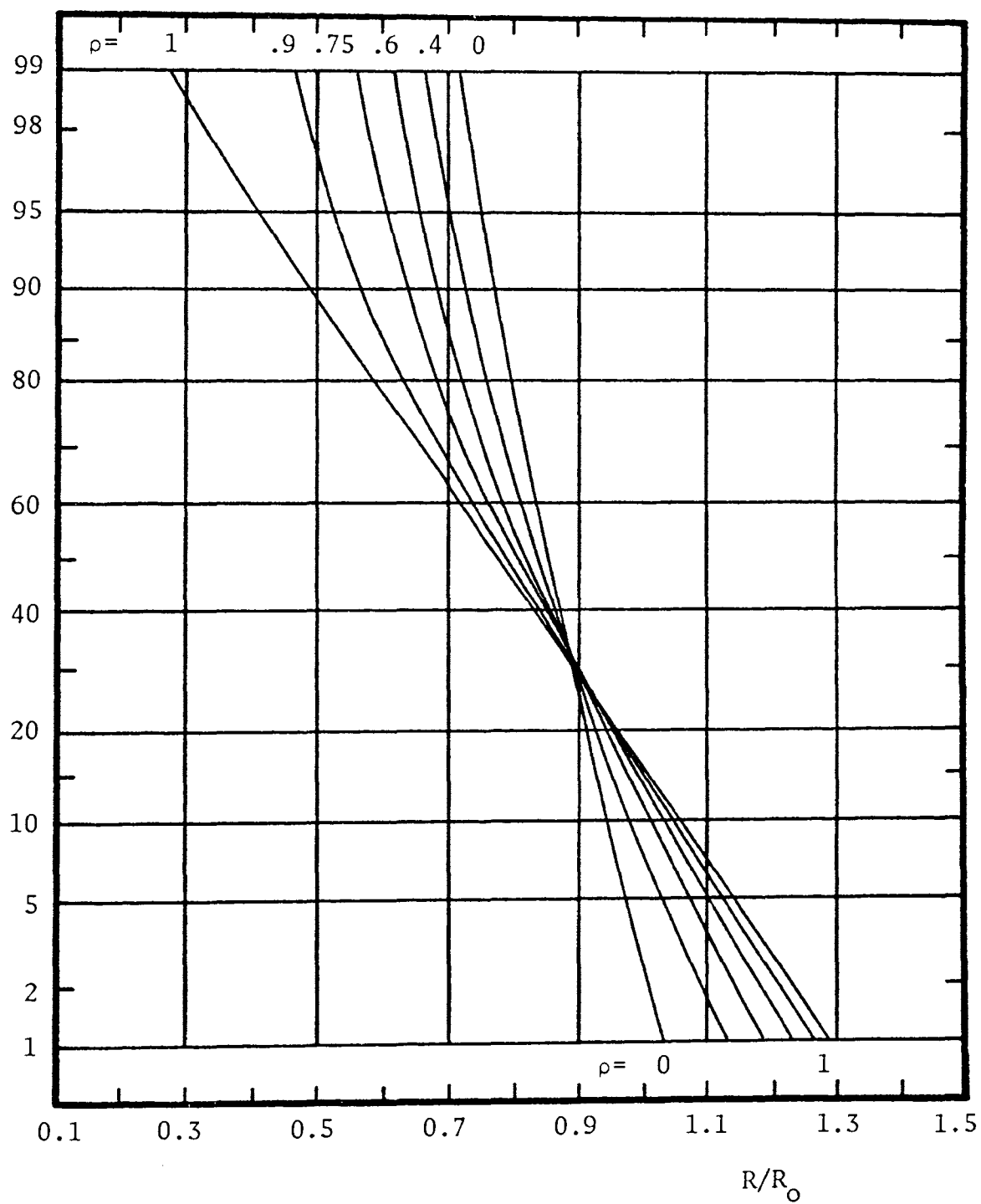
P_d in %


FIGURE 12

Probability of detection vs. R/R_0 for
 $N = 25$, probability of false alarm = 10^{-10}

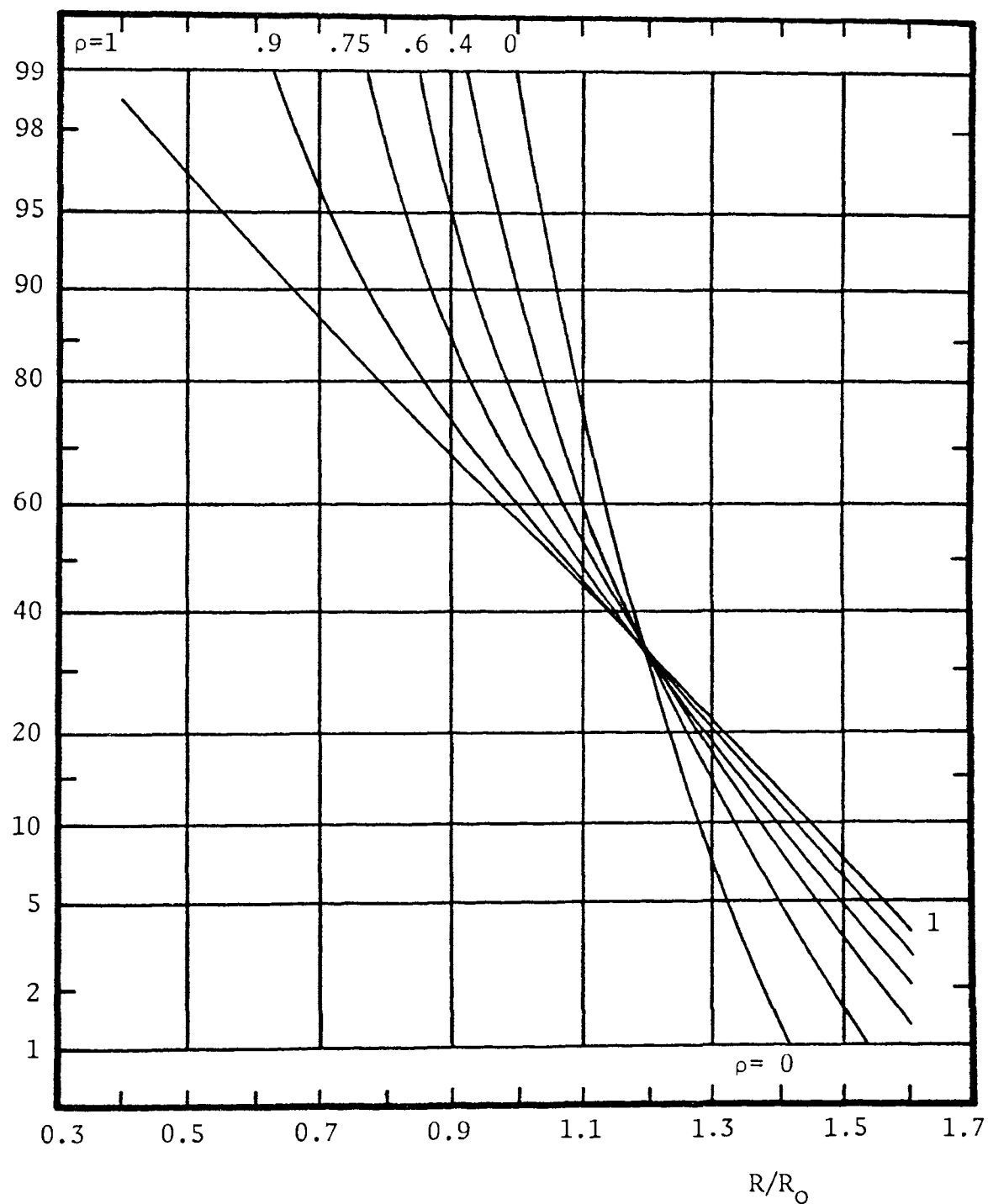
P_d in %


FIGURE 13

Probability of detection vs. R/R_0 for
 $N=100$, probability of false alarm = 10^{-6}

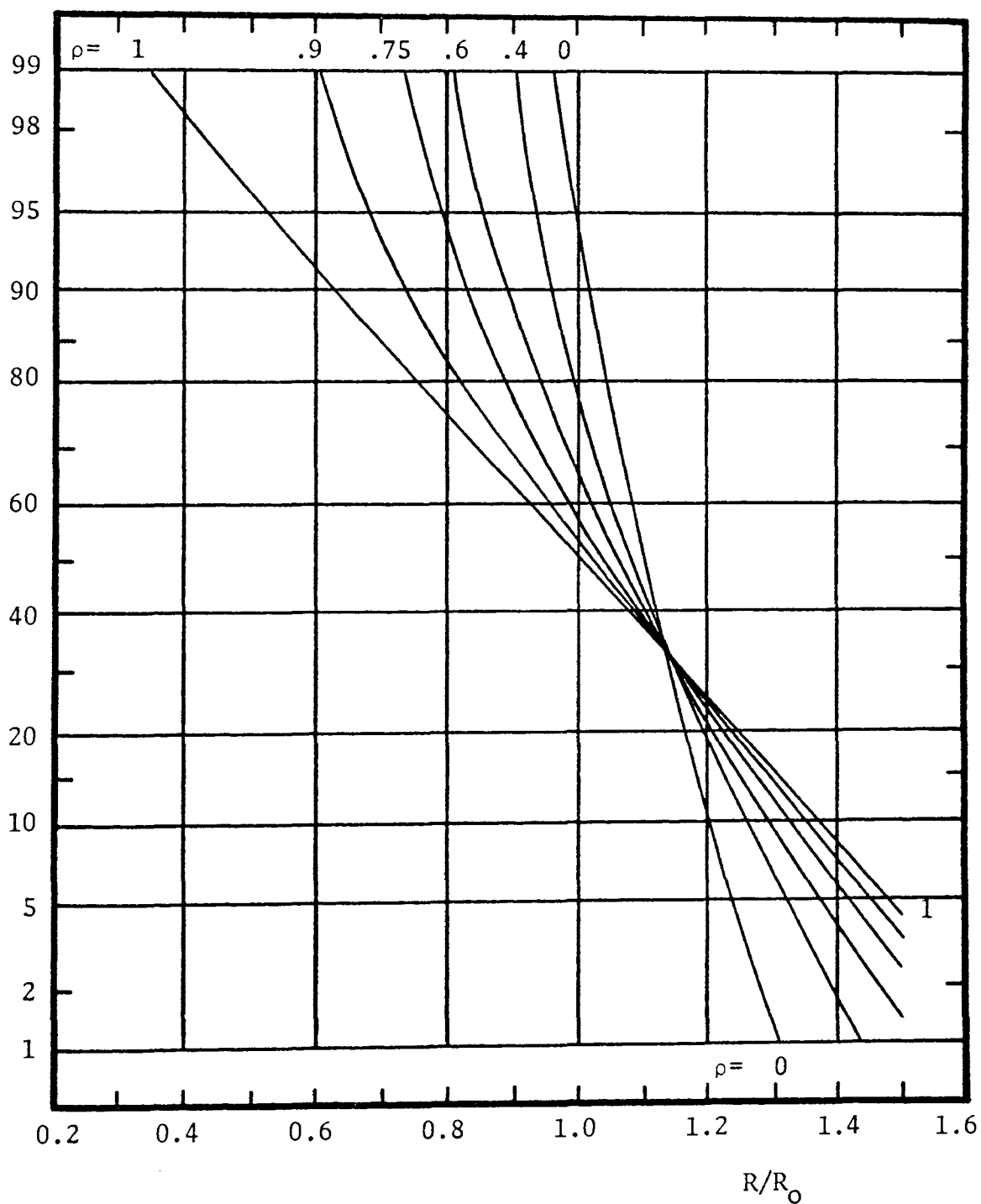
P_d in %


FIGURE 14

Probability of detection vs. R/R_0 for
 $N = 100$, probability of false alarm = 10^{-8}

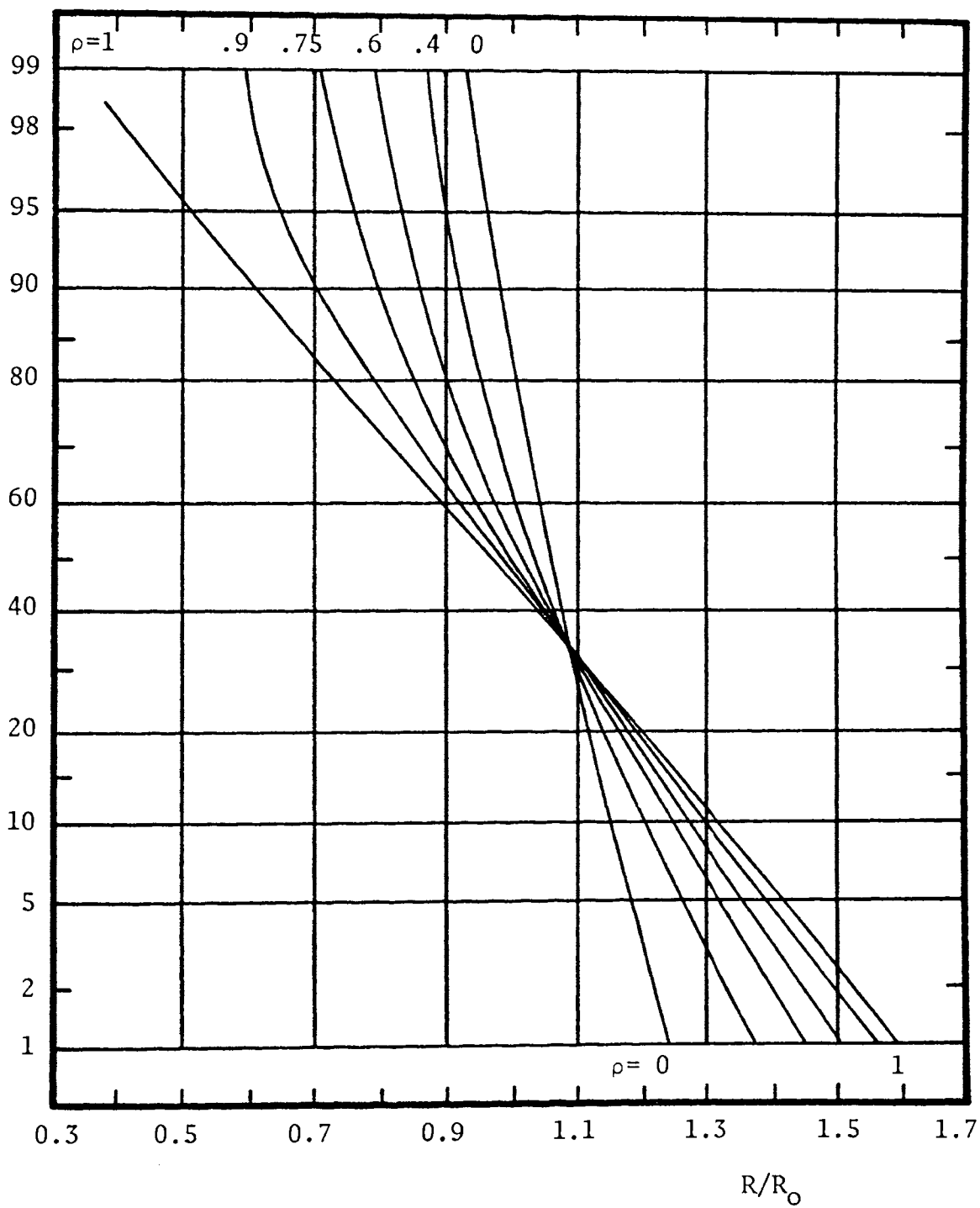
P_d in %


FIGURE 15

Probability of detection vs. R/R_0 for
 $N = 100$, probability of false alarm = 10^{-10}

gives the values of Y_b associated with each of the parameters N and P_f listed above.

		N		
		10	25	100
P_f	10^{-6}	32.71034	56.30405	154.9957
	10^{-8}	38.79901	63.85046	166.6777
	10^{-10}	44.62786	70.92190	177.3336

TABLE I

Bias level for $N = 10, 25,$ and 100 , probability of false alarm = $10^{-6}, 10^{-8},$ and 10^{-10}

C. Illustrative Example

In this section, the example presented in Chapter II will be re-examined. P_d is found from the curves presented in the previous section for the target described in Chapter II when a frequency agile transmitter is utilized. The transmitter frequency is a random variable whose PDF is described by the anti-sine distribution. The problem is restated as follows:

1. Target depth 5 meters.
2. Return fluctuations, for a fixed frequency transmitter, described by Swerling's Case I model.
3. 10 pulses integrated.
4. Transmitter frequency is an independent random variable for each pulse and is obtained from the anti-sine PDF.
5. Transmitter bandwidth (maximum possible excursion between pulses) 50 megahertz.
6. $P_f = 10^{-8}$
7. Find \bar{x} required for $P_d = 90\%$.

The solution of the problem is obtained as follows:

1. From Figure 6, $\rho = 0.43$, for this target depth and transmitter bandwidth.
2. From Figure 8, $R/R_0 = 0.63$ for $\rho = 0.43$, $P_f = 10^{-8}$, $N=10$, $P_d = 90\%$.
3. This gives a required \bar{x} of $(1/0.63)^4 = 6.35$.
4. Note that the solution in Chapter II requires an \bar{x} of only $(1/0.68)^4 = 4.7$ for $P_d = 90\%$, but the transmitter bandwidth required is 300 megahertz.
5. This is a difference of only 1.3db in transmitter power, and is more than canceled out by the bandwidth requirement, which is 6 times less when the anti-sine distribution is utilized.

CHAPTER V

DETECTION PROBABILITIES FOR SWERLING'S
MODIFIED CASE I TARGET

A. Description of Target

Experimental data has recently been published which indicate that Swerling's Case I model is not always an adequate description of the return fluctuations from a jet aircraft, for fixed frequency operation. (Swerling's Case I model assumes the amplitudes of the voltage returns are constant for the time on target during any one scan.) The experimental data indicate that these amplitudes are actually decorrelated with time.⁽⁷⁾ This implies that the returns in radars with low pulse repetition rates may, in reality, be partially decorrelated. The same statement may also apply for radars which integrate a sufficiently large number of pulses. With the theory discussed in this dissertation, it is a relatively simple matter (in theory) to calculate detection probabilities for targets whose returns are partially decorrelated due to the time difference between successive echos, both for fixed frequency and frequency agile radars.

Examination of the experimental results indicate that the correlation between the amplitudes of the voltage returns may be approximated by an exponential function. Since the correlation function was derived for the voltage returns from a target, it is obvious that the time decorrelation function and the correlation coefficient derived in Chapter III may be multiplied together to obtain the true correlation coefficient for the voltage returns. (The reflecting mechanism and the medium of propagation are both linear and therefore the homogeneity principle may be applied.) It will be

assumed that the time decorrelation function is described by (74), where τ_c is a fixed quantity for a particular set of circumstances. Experimental evidence indicates that τ_c may vary from 0.01 to 0.25 seconds, depending on the velocity and position of the target with respect to the radar.

$$R(t ; \tau_c) = \exp(-t/\tau_c) \quad (74)$$

The average correlation coefficient, ρ , is now modified by multiplying the ρ obtained in Chapter III by (74). This means that the correlation matrix, which is utilized in finding the characteristic function, is modified. Specifically, each off-diagonal term in the correlation matrix now becomes $\rho \exp(-|I-J|/\tau_c P_{rr})$, where I and J refer to the Ith row and Jth column of the matrix. τ_c is the correlation time constant defined in (74), P_{rr} is the pulse repetition frequency in hertz per second, and ρ is the value of the average correlation coefficient obtained in Chapter III for a frequency agile radar.

A simple closed form solution does not exist for a target of this type. It can only be written in general terms, and becomes quite complex if some of the eigenvalues of the modified correlation matrix are identical. If all of the eigenvalues are distinct, the solution may be obtained on a computer, for fixed parameters. The solution consists of finding the eigenvalues of the correlation matrix, substituting these into (45), finding the inverse transform of (45), and integrating the inverse transform from the bias level to infinity.

B. Calculation of P_d for Swerling's Modified Case I Model

The computer program in Appendix D was utilized to obtain curves of P_d versus R/R_0 for different values of the various parameters

involved. This is the first time this problem has been attacked, for N greater than two. (The results of Schwartz are sufficient for $N=2$. (5)) N was chosen as 10 and 25, because the accuracy of the computer supplied sub-routine which calculates the eigenvalues becomes suspect for larger matrices. P_f was chosen equal to 10^{-6} and 10^{-10} in order to reduce the total number of Figures. (Also, one may easily interpolate between the two values of P_f if desired.) The product of τ_c and P_{rr} was chosen to be representative of typical radars.

Figures 16 through 18 present P_d versus R/R_0 for $N=10$, $P_f=10^{-6}$, various values of ρ , the correlation coefficient found in Chapter III for a frequency agile radar without time decorrelation, and the product of τ_c and P_{rr} equal to 10, 25, and 100. Figures 19 through 21 present the same curves with $P_f=10^{-10}$. When the product of τ_c (in seconds) and P_{rr} (in hertz per second) is greater than 100 hertz, the time decorrelation factor may be neglected and P_d may be obtained from the curves in Chapter IV.

Curves of P_d versus R/R_0 are presented in Figures 22 and 23 for $N=25$, and $P_f=10^{-6}$ and 10^{-10} , respectively. Note that ρ , the correlation coefficient of Chapter III, is unity for all curves in these two figures. Unfortunately, the computer program of Appendix D did not converge for ρ less than 0.9 when N was 25. However, when the product of τ_c and P_{rr} is greater than 200 hertz, the curves of Chapter IV may again be used, for $N=25$. One may interpolate between the curves of Figures 22 and 23 for values of ρ between zero and one, with fair accuracy. It was felt that the presentation of the general theory with complete results for $N=10$ and limited results for $N=25$ was sufficient, since computer programming was not the prime objective of this dissertation.

Actually, the most important information in this chapter is contained in the curves for $\rho=0$ and $\rho=1$. These two curves indicate the maximum improvement in detection range that can be accrued by use of frequency agility. This can be compared to the maximum improvement found in the previous chapter, where all pulses were assumed to be completely correlated prior to use of frequency agility. From the results presented, it can be seen that most of the possible improvement in detection range is achieved if ρ is less than 0.4.

The curves of this chapter may be employed in the same manner as those in Chapter IV. An added parameter, the product of τ_c and P_{rr} , has been introduced. Otherwise, the curves may be utilized in the same manner as those in Chapter IV. For this reason, an illustrative example will not be presented.

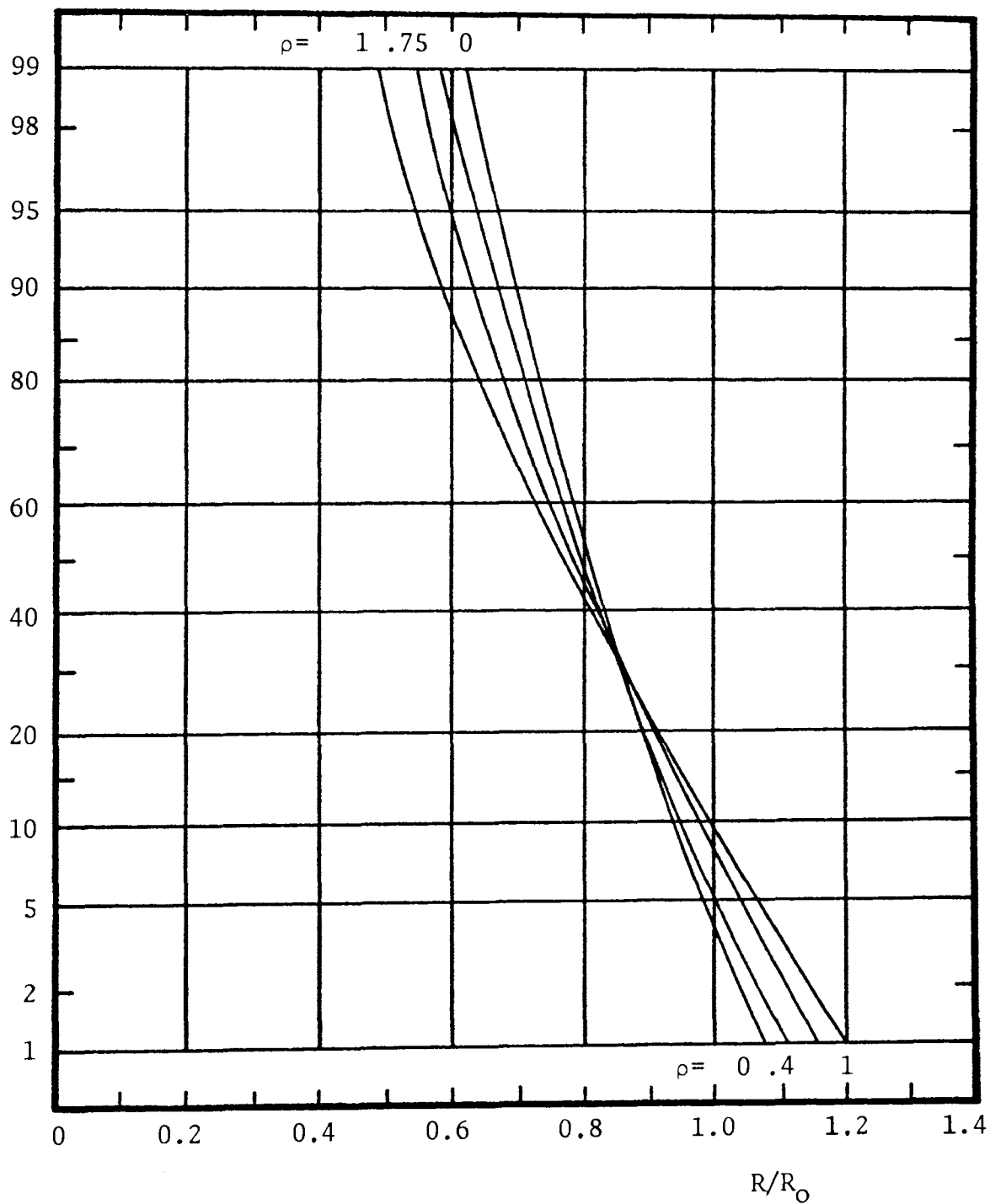
P_d in %


FIGURE 16

Probability of detection vs. R/R_0 for
 $N=10$, probability of false alarm = 10^{-6} ,
 product of τ_c and $P_{rr} = 10$

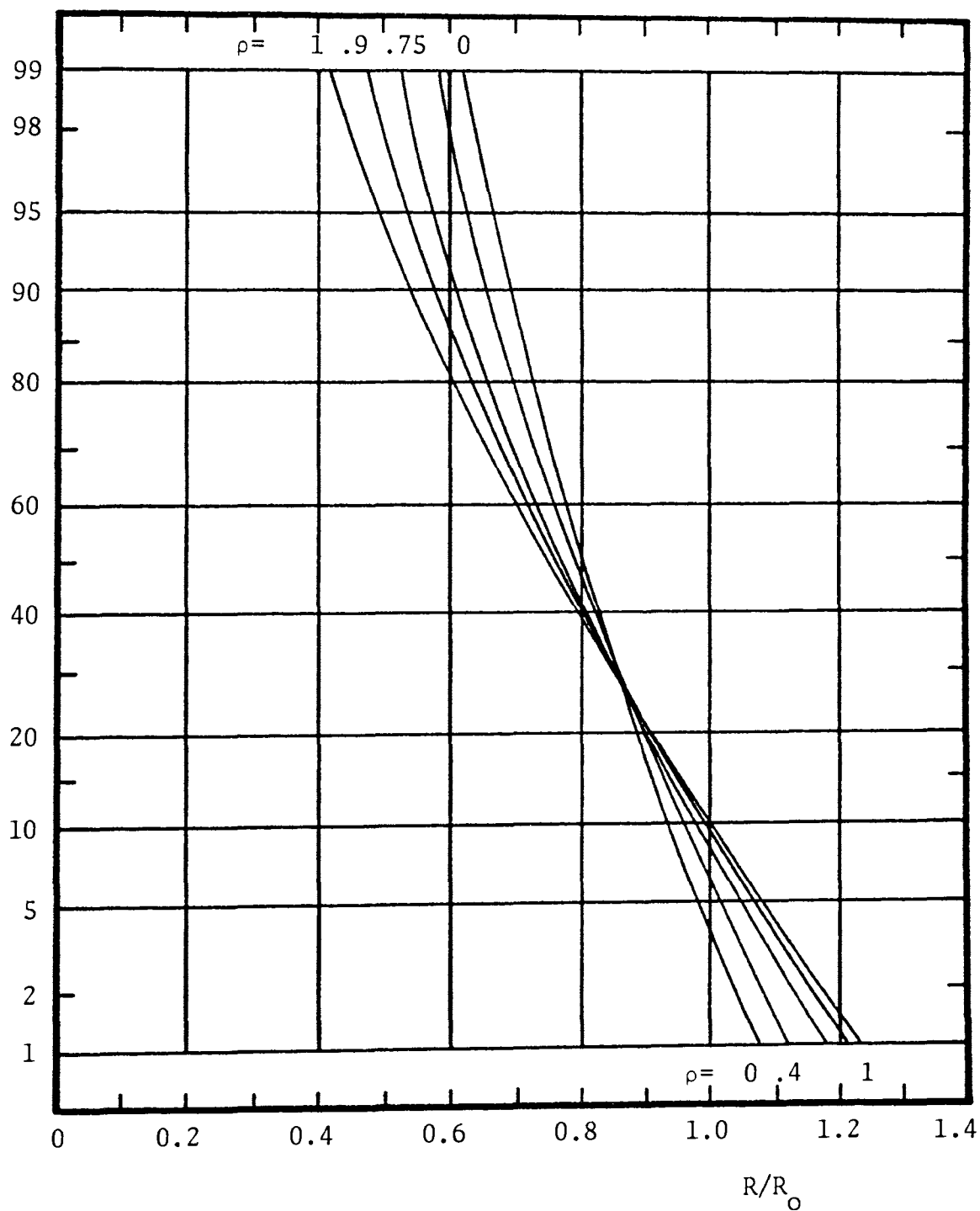
P_d in %


FIGURE 17

Probability of detection vs. R/R_0 for $N=10$, probability of false alarm = 10^{-6} , product of τ_c and $P_{rr} = 25$

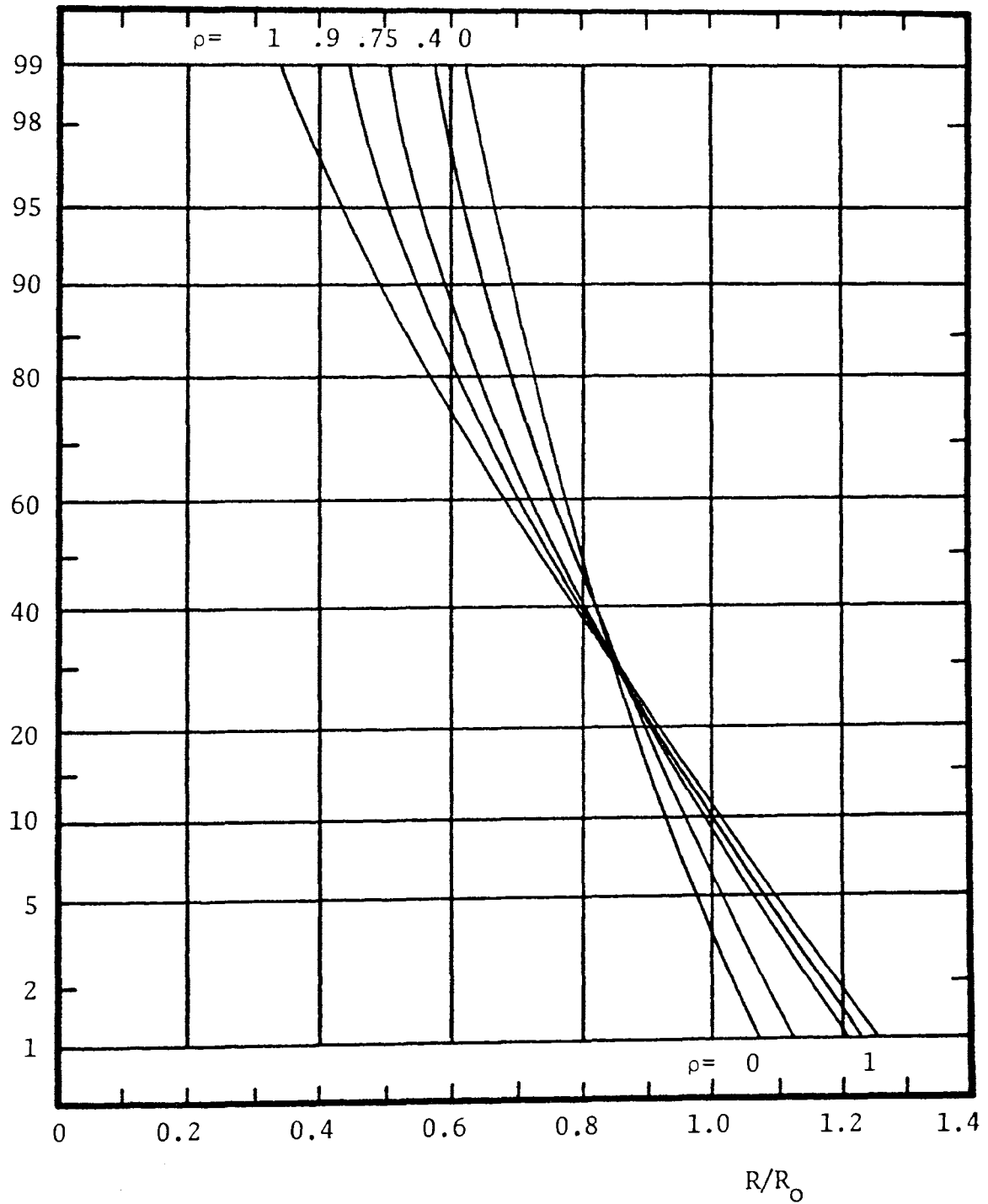
P_d in %


FIGURE 18

Probability of detection vs. R/R_0 for $N=10$, probability of false alarm = 10^{-6} , product of τ_c and $P_{rr} = 100$

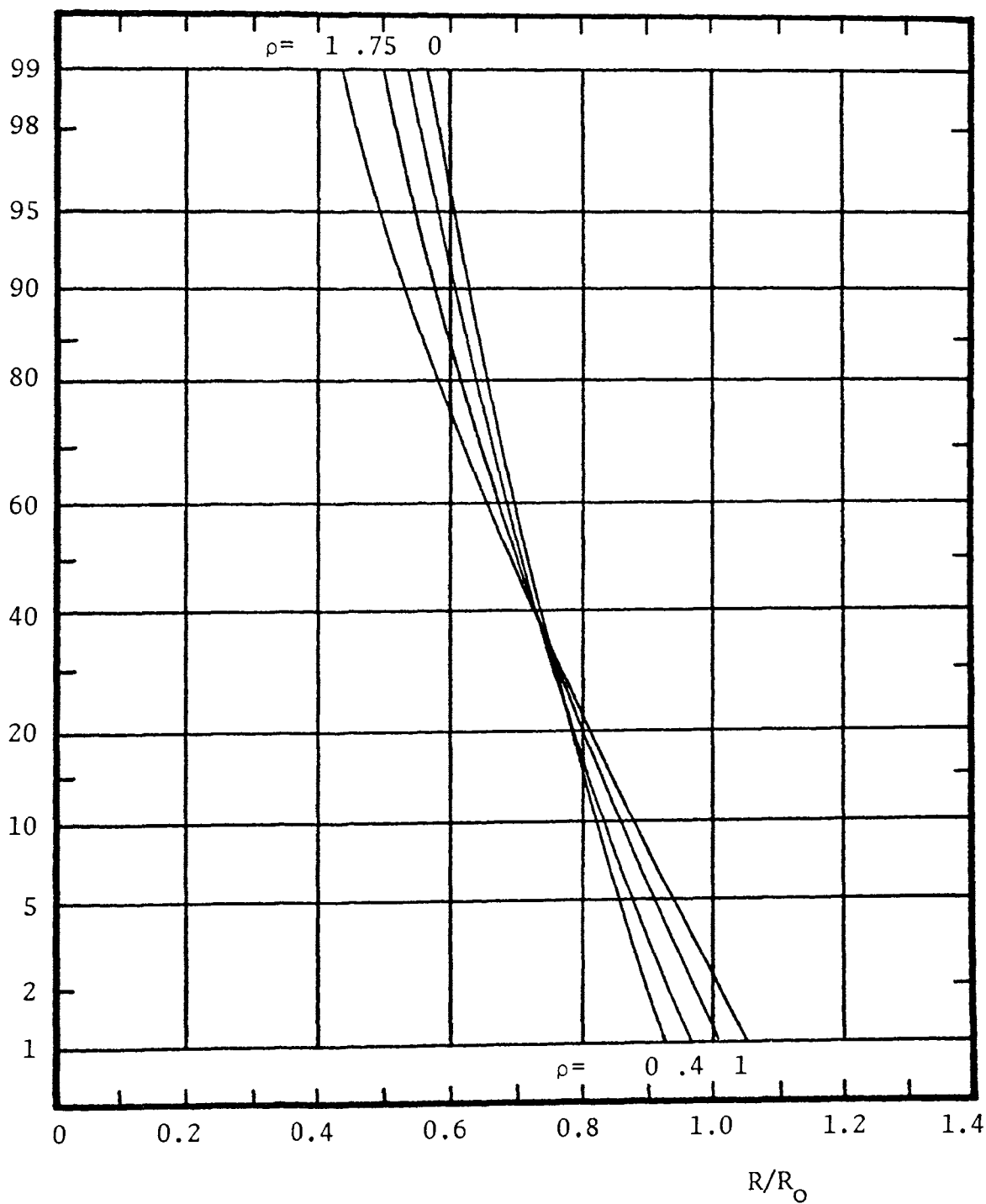
P_d in %


FIGURE 19

Probability of detection vs. R/R_0 for $N=10$, probability of false alarm = 10^{-10} , product of τ_c and $P_{rr} = 10$

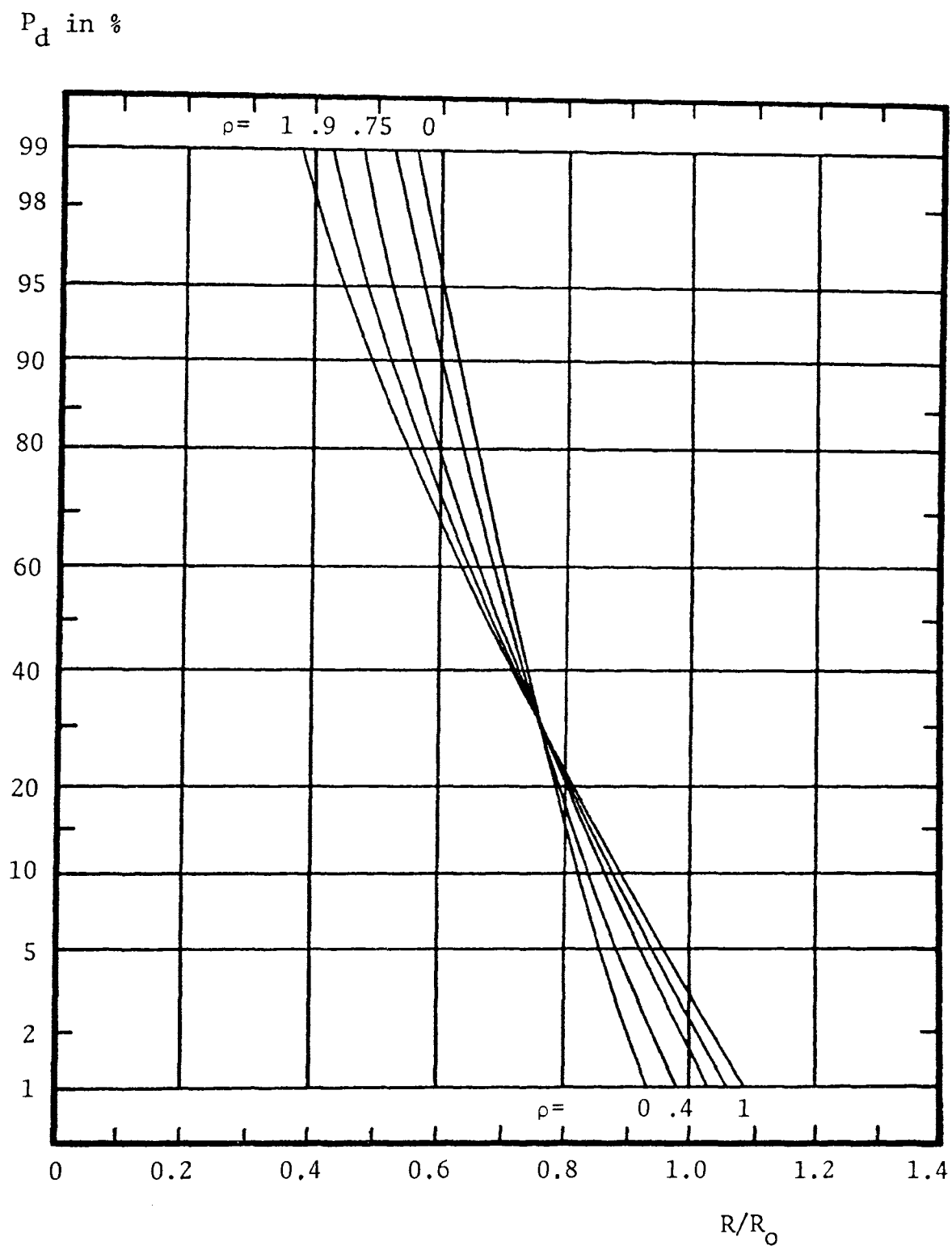


FIGURE 20

Probability of detection vs. R/R_0 for $N=10$, probability of false alarm = 10^{-10} , product of τ_c and $P_{rr} = 25$

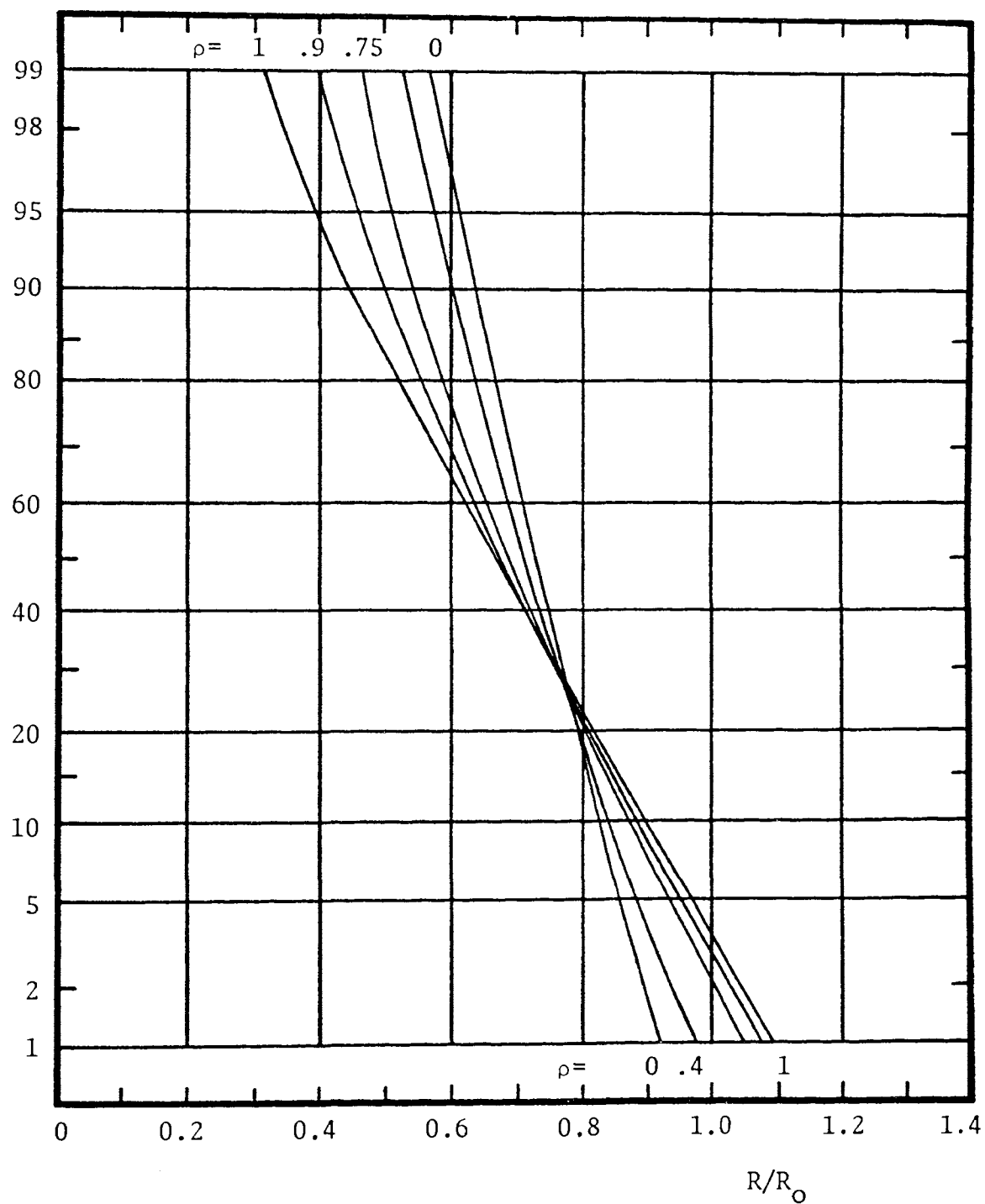
P_d in %


FIGURE 21

Probability of detection vs. R/R_0 for $N=10$, probability of false alarm = 10^{-10} , product of τ_c and $P_{rr} = 100$

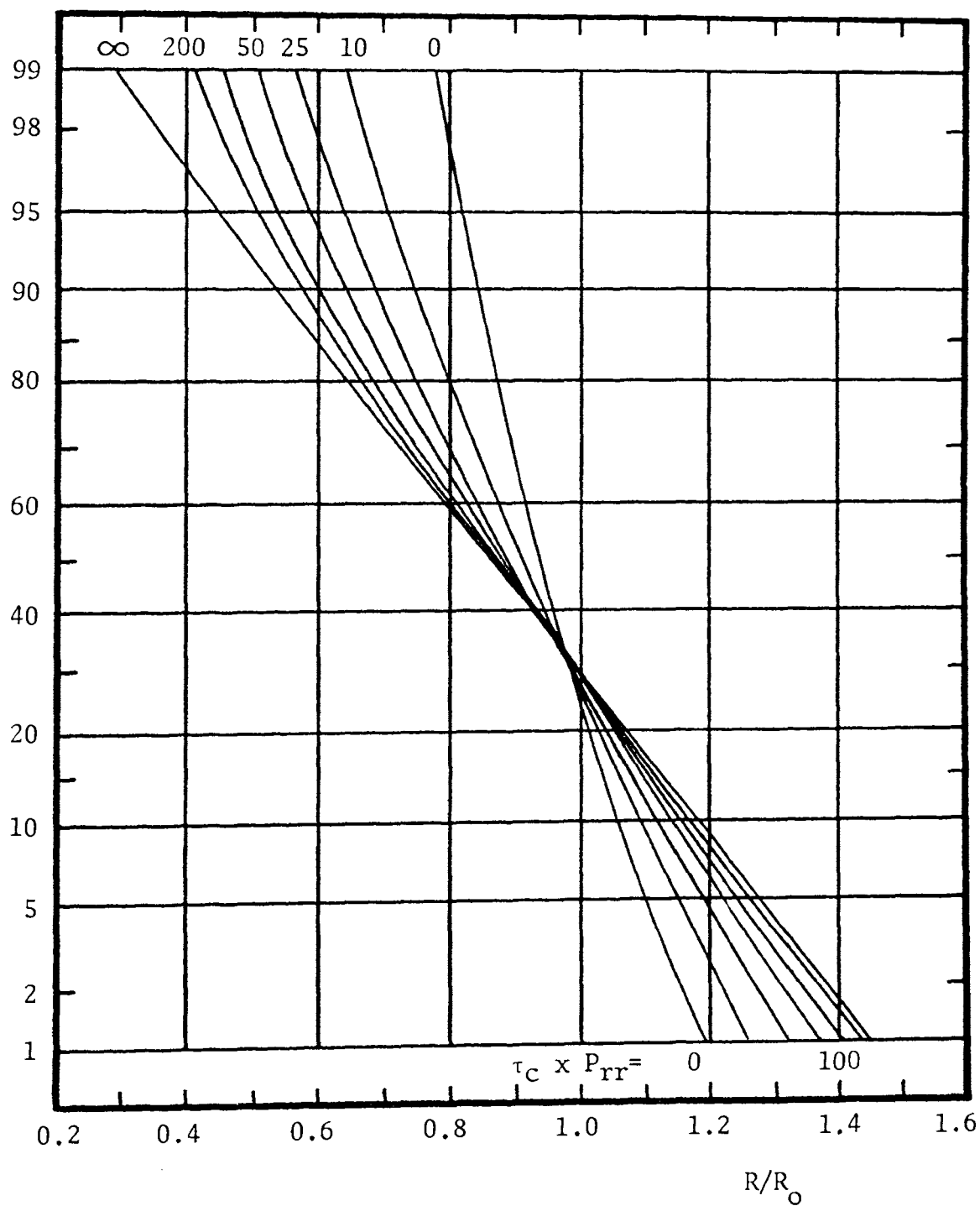
P_d in %


FIGURE 22

Probability of detection vs. R/R_0 for $N=25$, probability of false alarm = 10^{-6} , correlation coefficient = 1 in all cases

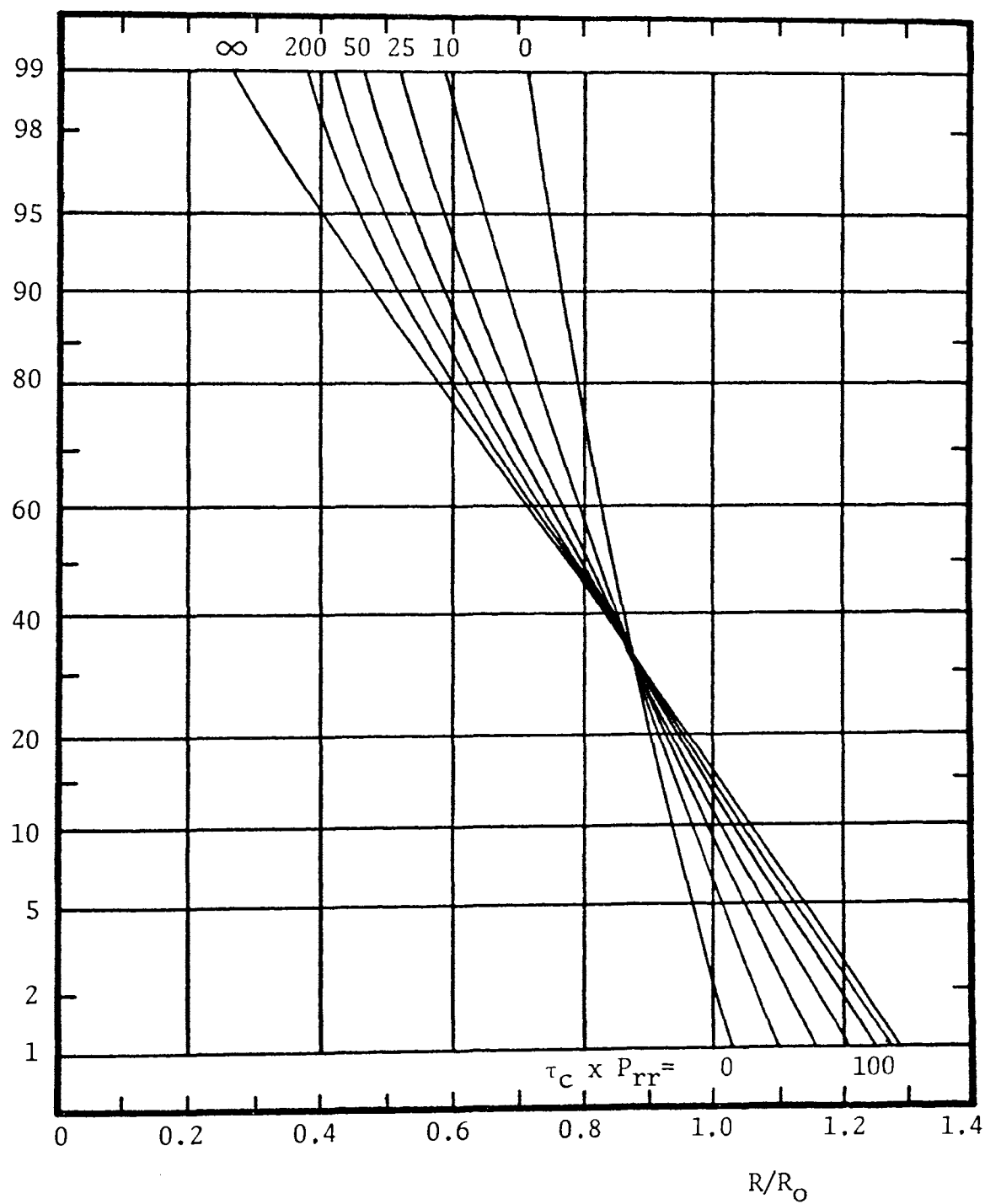
P_d in %


FIGURE 23

Probability of detection vs. R/R_0 for $N=25$, probability of false alarm = 10^{-10} , correlation coefficient = 1 in all cases

CHAPTER VI

CONCLUSIONS AND SUGGESTIONS FOR FURTHER INVESTIGATIONS

A. Conclusions

Detection probabilities have been calculated for a pulse non-coherent radar with a frequency agile transmitter. In Chapter IV it was assumed that the target return fluctuations could be described by Swerling's Case I model, while the results in Chapter V apply to the same target, modified by an exponential time decorrelation function. The frequency of the transmitter is described by probability density functions in this dissertation. It has been shown that a radar of this type (random frequency for each pulse) requires considerably less bandwidth than a radar with monotonic frequency changes (non-random) between pulses, when all parameters are held fixed, with a small decrease in detection range. The equations required for the calculation of the detection probabilities were derived. It is anticipated that the results of this dissertation will find direct applications in future non-coherent frequency agile pulsed radars.

The correlation coefficient (off-diagonal terms in the correlation, or normalized covariance, matrix) was derived for two practical types of radar transmitters. Specifically, transmitters whose frequency is described by the uniform and anti-sine probability density functions. It should again be noted that the curves for P_d in Chapter IV and V are general in that they apply for any non-coherent pulsed frequency agile radar whose frequency can only be described by a probability density function, provided the correlation coefficient can be calculated. An additional factor was introduced in the correlation matrix to make it conform more closely to experimental results in Chapter V. The

general computer program required to calculate P_d under these constraints was developed and is included in Appendix D.

B. Suggestions for further Investigations

The next area of investigation should probably be the development of the theory required to calculate detection probabilities for Swerling's Case III and IV models, for frequency agile radars. Targets of this type are characterized by one dominant reflector and many small ones, such as satellites and ships. Detection of ships is complicated by sea clutter. The characteristic function of the integrator output will be similar to (45), and again will depend on the eigenvalues of the correlation matrix.

An additional area of investigation is the application of random frequency agility to the tracking problem. It is anticipated that a reduction in transmitter bandwidth will also be feasible in this application.

The third area that should receive attention is the computer program in Appendix D. A program could be written that will allow an approximate solution when the eigenvalues of the correlation matrix are too close together to allow the present program to converge. This, coupled with a generalized program which finds the inverse transform of the characteristic function when m of the eigenvalues are repeated n_m times, would be sufficient to solve the convergence problem in the program of Appendix D.

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APPENDIX A - DERIVATION OF EQUATION 7

The probability distribution of the envelope of the output of a narrow band filter, with a sinusoidal signal whose frequency is in the passband of the filter and additive white Gaussian noise present at the input to the filter, is derived in this appendix. This is included for completeness, but no claim of originality is made by the author. The notation chosen was a combination of that utilized by Lawson and Uhlenbeck in Vol. 24 of the Radiation Laboratory Series, Threshold Signals ⁽⁹⁾, Davenport and Root in Random Signals and Noise ⁽¹⁷⁾, Berkowitz in Modern Radar ⁽¹²⁾, and a host of other references too numerous to mention. In addition, it was desired that the answer correspond to equation (7) in the first chapter.

The amplitude of the noise voltage at the input to the filter can be expanded in a Fourier series if we assume the voltage has a period T. (T will eventually be allowed to approach infinity). The series may be written as

$$n_{in}(t) = \sum_{k=0}^{\infty} a_k \cos 2\pi k f_0 t + b_k \sin 2\pi k f_0 t \quad (A1)$$

where the a_k and b_k are independent Gaussian random variables (R.V.) with

$$\begin{aligned} E(a_k) &= E(b_k) = 0 & E(a_k a_j) &= E(b_k b_j) = \sigma^2 \delta_{kj} / T \\ E(a_k b_j) &= 0 & f_0 &= 1/T \end{aligned} \quad (A2)$$

δ_{kj} is the Kroneker del which equals one when $k=j$ and zero otherwise. $\sigma^2 = k T_r \bar{N}F$ where this k is Boltzmann's constant, T_r is the absolute temperature of the receiver, and $\bar{N}F$ is the overall noise figure of the receiver. This is also the average input noise power per cycle. After this noise signal passes through the filter, the

output noise voltage may be written as

$$n_{\text{out}}(t) = \sum_{k=0}^{\infty} A_k [a_k \cos(2\pi k f_o t - \phi_k) + b_k \sin(2\pi k f_o t - \phi_k)] \quad (\text{A3})$$

The signal voltage out of the filter may be written as

$$s_{\text{out}}(t) = P(t) \sin(2\pi f_c t + \theta) \quad (\text{A4})$$

where f_c is the carrier frequency.

The sum of the signal and noise voltages at the output of the filter may now be written as

$$\begin{aligned} s_{\text{out}}(t) + n_{\text{out}}(t) &= P(t) \cos\theta \sin 2\pi f_c t \\ &+ \sum_{k=0}^{\infty} A_k [a_k \cos(2\pi k f_o t - \phi_k - 2\pi f_c t + 2\pi f_c t) \\ &+ b_k \sin(2\pi k f_o t - \phi_k - 2\pi f_c t + 2\pi f_c t)] \end{aligned} \quad (\text{A5})$$

After expanding the terms in the summation as indicated by the underbars, (A5) may be written as

$$s_{\text{out}}(t) + n_{\text{out}}(t) = X \cos 2\pi f_c t + Y \sin 2\pi f_c t \quad (\text{A6})$$

where

$$\begin{aligned} X &= P(t) \sin\theta + \sum_{k=0}^{\infty} A_k [a_k \cos(2\pi k f_o t - 2\pi f_c t - \phi_k) \\ &+ b_k \sin(2\pi k f_o t - 2\pi f_c t - \phi_k)] \end{aligned} \quad (\text{A7})$$

$$Y = P(t) \cos \theta + \sum_{k=0}^{\infty} A_k [-a_k \sin(2\pi k f_o t - 2\pi f_c t - \phi_k)] + b_k \cos(2\pi k f_o t - 2\pi f_c t - \phi_k)] \quad (A8)$$

Inspection of (A6) indicates that the envelope of the filter output will be given by

$$R(t) = (X^2 + Y^2)^{1/2} \quad (A9)$$

From (A7) and (A8), X and Y are linear combinations of the Gaussian R.V. a_k and b_k and are therefore Gaussian R.V. themselves. Their means are $P(t) \sin \theta$ and $P(t) \cos \theta$, respectively. The variance of X is equal that of Y and is given by (A10).

$$\sigma_X^2 = E \left\{ [X - P(t) \sin \theta]^2 \right\} = E \left\{ \left[\sum_{k=0}^{\infty} A_k (a_k \cos z + b_k \sin z) \right]^2 \right\} \quad (A10)$$

where $z = 2\pi k f_o t - 2\pi f_c t - \phi_k$

Due to conditions (A2), (A10) reduces to (A11).

$$\sigma_X^2 = \frac{\sigma^2}{T} \sum_{k=0}^{\infty} A_k^2 \quad (A11)$$

The A_k are the voltage transfer coefficients of the filter at the frequencies $k f_o$. Now if T, the period of the noise voltage, approaches infinity, (A11) becomes

$$\sigma_X^2 = \lim_{T \rightarrow \infty} \frac{\sigma^2}{T} \sum_{k=0}^{\infty} A_k^2 = \sigma^2 \int_0^{\infty} A^2(f) df \quad (A12)$$

Since $A(f)$ is the transfer function of the filter, σ_X^2 becomes the

total average noise power at the filter output. This will be called ψ_0 henceforth.

After calculating the correlation between X and Y (which is zero), it is obvious that X and Y are independent Gaussian R.V. with means as previously stated and variance ψ_0 . Therefore, the joint distribution of X and Y is

$$f(X,Y) = \frac{1}{2\pi\psi_0} \exp \left\{ -\frac{1}{2\psi_0} [X-P \sin \theta]^2 + (Y-P \cos \theta)^2 \right\} \quad (A13)$$

where the argument of the function P(t) has been omitted for brevity. Transforming (A13) to polar coordinates through (A14), (A15) results.

$$x = R \cos \phi \quad Y = R \sin \phi \quad |J| = R \quad (A14)$$

$$f(R,\phi) = \frac{R}{2\pi\psi_0} \exp \left\{ -\frac{1}{2\psi_0} (R^2+P^2 - 2PR \sin(\phi-\theta)) \right\} \quad R \geq 0 \quad (A15)$$

Because of the form in which f(X,Y) was originally written, the PDF f(R) is found by assuming $\phi-\theta$ is a uniformly distributed R.V. over 0, 2π . The difference is only a constant which is immaterial because the final PDF would need re-normalization if this assumption were not made. f(R) is found through (A16).

$$f(R) = \frac{R}{\psi_0} \exp \left[-\frac{1}{2\psi_0} (R^2+P^2) \right] \int_0^{2\pi} \exp(PR \sin \theta/\psi_0) \frac{d\theta}{2\pi} \quad R \geq 0 \quad (A16)$$

The integral in (A16) is well known. The general form is

$$\int_0^{2\pi} \exp (a \cos \theta + b \sin \theta) \frac{d\theta}{2\pi} = I_0 (a^2+b^2)^{1/2}$$

Where I_0 represents the modified Bessel function of the first kind

of order zero. Therefore, (A16) becomes

$$f(R) = \frac{R}{\psi_0} \exp \left[-\frac{1}{2\psi_0} (R^2 + P^2) \right] I_0(PR/\psi_0) \quad R \geq 0 \quad (A17)$$

For the case where noise only is present at the filter input ($P=0$), (A17) reduces to

$$f(R) = \frac{R}{\psi_0} \exp \left[-\frac{R^2}{2\psi_0} \right] \quad R \geq 0 \quad (A18)$$

This concludes the development.

APPENDIX B - DERIVATION OF EQUATION 25

The correlation function relating the amplitudes of the pulsed voltage returns from a target is derived in this section. The received pulses have a known fixed frequency shift between pulses. This analysis neglects any doppler shift associated with relative target movement. This assumption is valid for practical target velocities and carrier frequencies in the microwave region.

Assume the target consists of m independent randomly spaced scatterers, uniformly spread over a radial distance $r - D/2$ to $r + D/2$ from the radar. Further, assume N pulses strike the target. The voltage return from the i th pulse, $i = 1, 2, \dots, N$, may be written as

$$V_{k,i}(f) = \sum_{q=1}^m a_q \sin(4\pi f r_q / c + \theta_q) \quad (B1)$$

f is the frequency of the i th transmitter pulse, c is the speed of light in air, θ_q is an independent random variable assumed to be uniformly distributed from 0 to 2π , and r_q is the radial distance (normal to the antenna) from the mean center of the target to the q th scatterer. Because θ_q is uniformly distributed, $E[\sin(4\pi f r_q / c + \theta_q)]$ equals zero. From the central limit theorem, it is easily seen that $V_{k,i}(f)$ is normally distributed with mean zero and variance equal to the sum of the variances of the m terms, $a_q^2 \sin^2(4\pi f r_q / c + \theta_q)$.

$$V_{k,i}(f) \sim N(0, \sigma^2) \quad (B2)$$

where

$$\sigma^2 = E \left\{ \sum_{q=1}^m [a_q \sin(4\pi f r_q / c + \theta_q)]^2 \right\} \quad (B3)$$

The correlation function relating any two voltage returns at frequencies f_i and f_j may be written as

$$R(f_i - f_j) = \frac{E[V_{k,i}(f_i) V_{k,j}(f_j)]}{\{\text{Var}[V_{k,i}(f_i)] \text{Var}[V_{k,j}(f_j)]\}^{1/2}} \quad (B4)$$

The expectation in (B4) may be written as

$$E[V_{k,i}(f_i) V_{k,j}(f_j)] = E \left\{ \sum_{p=1}^m \sum_{q=1}^m a_q a_p [\sin(4\pi f_i r_q / c + \theta_q) \sin(4\pi f_j r_p / c + \theta_p)] \right\} \quad (B5)$$

Since θ_q and θ_p are random independent uniformly distributed variables over the range of $0, 2\pi$, (B5) reduces to (B6) after taking the expectation over both θ_q and θ_p .

$$E[V_{k,i}(f_i) V_{k,j}(f_j)] = E \left\{ \sum_{q=1}^m \frac{a_q^2}{2} \cos[(4\pi r_q (f_i - f_j) / c)] \right\} \quad (B6)$$

From (B3),

$$\begin{aligned} \sigma^2 &= E \left[\sum_{q=1}^m a_q^2 \sin^2(4\pi f r_q / c + \theta_q) \right] \\ &= \sum_{q=1}^m a_q^2 E[\sin^2(4\pi f r_q / c + \theta_q)] = \sum_{q=1}^m \frac{a_q^2}{2} \end{aligned} \quad (B7)$$

Substitution of (B7) and (B6) into (B4) yields (B8)

$$R(f_i - f_j) = \frac{E \left\{ \sum_{q=1}^m \frac{a_q^2}{2} \cos[4\pi r_q (f_i - f_j)/c] \right\}}{\sum_{q=1}^m \frac{a_q^2}{2}} \quad (B8)$$

The expectation in (B8) is to be taken with respect to r_q . It has already been assumed that r_q is uniformly distributed from $-D/2$ to $+D/2$. Therefore, the numerator of (B8) becomes

$$\begin{aligned} & \int_{-D/2}^{D/2} \sum_{q=1}^m \frac{a_q^2}{2} \cos[4\pi r_q (f_i - f_j)/c] \cdot \frac{1}{D} dr_q \\ &= \sum_{q=1}^m \frac{a_q^2}{2} \frac{\sin[2\pi D(f_i - f_j)/c]}{2\pi D(f_i - f_j)/c} \end{aligned} \quad (B9)$$

Substitution of (B9) into (B8) yields the correlation function relating the amplitudes of the voltage returns, (B10).

$$R(f_i - f_j ; D) = \frac{\sin[2\pi D(f_i - f_j)/c]}{2\pi D(f_i - f_j)/c} \quad (B10)$$

This is equation (25) in Chapter II, and is also known as the normalized covariance function.

APPENDIX C - DERIVATION OF EQUATION 68

The eigenvalues for an $N \times N$ matrix, which has all off-diagonal terms equal to ρ , $0 \leq \rho \leq 1$, and all diagonal elements equal to unity, are found in this appendix. The matrix under consideration is given in (C1). It is postulated that the eigenvalues are given by (C2), which is (68) in Chapter IV.

$$\begin{bmatrix}
 1 & \rho & \rho & \cdot & \cdot & \cdot & \rho \\
 \rho & 1 & \rho & \cdot & \cdot & \cdot & \rho \\
 \rho & \rho & 1 & \cdot & \cdot & \cdot & \rho \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \rho & \rho & \rho & \cdot & \cdot & \cdot & 1
 \end{bmatrix}
 \quad (C1)$$

$N \times N$

$$\begin{aligned}
 \lambda_1 &= 1 + (N-1)\rho \\
 \lambda_2 &= \lambda_3 = \dots = \lambda_N = 1 - \rho
 \end{aligned}
 \quad (C2)$$

The proof that (C2) gives the eigenvalues of (C1) is made by induction. It is easily seen that (C2) is true for $N=2$ and $N=3$. Now assume (C2) holds for $N=N$. The determinant which gives the eigenvalues may be written as

$$\begin{vmatrix}
 1-\lambda & \rho & \rho & . & . & . & \rho \\
 \rho & 1-\lambda & \rho & . & . & . & \rho \\
 \rho & \rho & 1-\lambda & . & . & . & \rho \\
 . & . & . & . & . & . & . \\
 . & . & . & . & . & . & . \\
 . & . & . & . & . & . & . \\
 \rho & \rho & \rho & . & . & . & 1-\lambda
 \end{vmatrix} = 0 \quad (C3)$$

N x N

Subtracting the first column of (C3) from the last yields (C4).

$$\begin{vmatrix}
 1-\lambda & \rho & \rho & . & . & . & \rho-(1-\lambda) \\
 \rho & 1-\lambda & \rho & . & . & . & 0 \\
 \rho & \rho & 1-\lambda & . & . & . & 0 \\
 . & . & . & . & . & . & . \\
 . & . & . & . & . & . & . \\
 . & . & . & . & . & . & 0 \\
 \rho & \rho & \rho & . & . & . & 1-\lambda-\rho
 \end{vmatrix} = 0 \quad (C4)$$

N x N

Now expand (C4) about the last column.

$$(-1)^{N+1} [\rho-(1-\lambda)] \begin{vmatrix}
 \rho & 1-\lambda & \rho & . & . & . & \rho \\
 \rho & \rho & 1-\lambda & . & . & . & \rho \\
 . & . & . & . & . & . & . \\
 . & . & . & . & . & . & . \\
 . & . & . & . & . & . & 1-\lambda \\
 \rho & \rho & \rho & . & . & . & \rho
 \end{vmatrix}$$

(N-1) x (N-1)

$$+(-1)^{2N} (1-\lambda-\rho) \begin{vmatrix} 1-\lambda & \rho & . & . & . & \rho \\ \rho & 1-\lambda & . & . & . & \rho \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \rho & \rho & . & . & . & 1-\lambda \end{vmatrix} = 0 \quad (C5)$$

(N-1) x (N-1)

The first determinant in (C5) is expanded as follows: Subtract column two from column one and expand about column 1. Repeat this procedure with the resulting determinant until a 2 x 2 determinant is left. The result is

$$\begin{vmatrix} \rho & 1-\lambda & \rho & . & . & . & \rho \\ \rho & \rho & 1-\lambda & . & . & . & \rho \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & 1-\lambda \\ \rho & \rho & \rho & . & . & . & \rho \end{vmatrix} =$$

(N-1) x (N-1)

$$[\rho - (1-\lambda)]^{N-1-2} \begin{vmatrix} \rho & 1-\lambda \\ \rho & \rho \end{vmatrix} = \rho [\rho - (1-\lambda)]^{N-2} \quad (C6)$$

Observe that the second determinant in (C5) is of the same form as (C3), except that it is of order (N-1) x (N-1) instead of N x N. Therefore, the solutions for λ are $\lambda = 1 + (N-2)\rho$ and N-2 solutions $\lambda = 1-\rho$. This follows from the assumption that (C2) is a solution for the eigenvalues of (C1) for any N.

Substituting these solutions and (C6) into (C5), (C7) is obtained.

$$\begin{aligned}
 & (-1)^{N+1} [\rho - (1-\lambda)]^\rho [\rho - (1-\lambda)]^{N-2} \\
 & + (-1)^{2N} (1-\rho-\lambda)^{N-2} [1+(N-2)\rho-\lambda](1-\rho-\lambda) = 0
 \end{aligned} \tag{C7}$$

Combining terms and factoring a minus one out of the first term yields (C8).

$$\begin{aligned}
 & (-1)^{N+1} (-1)^{N-1} \rho (1-\rho-\lambda)^{N-1} \\
 & + (-1)^{2N} (1-\rho-\lambda)^{N-1} [1+(N-2)\rho-\lambda] = 0
 \end{aligned} \tag{C8}$$

Obviously the minus one's raised to the power $2N$ are always positive. Therefore, (C8) becomes (C9) when the common term is factored out.

$$(1-\rho-\lambda)^{N-1} [1+(N-2)\rho-\lambda+\rho] = 0 \tag{C9}$$

There are N solutions for λ in (C9). By inspection, they are identical with (C2), and (68) of Chapter IV. Since (C9) holds for any N , the eigenvalues of (C1) are given by (C2), as was postulated.

APPENDIX D - COMPUTER PROGRAM

The computer program for calculating the detection probabilities for Swerling's Case I target, modified by the time decorrelation function in Chapter V, is presented in this appendix. The program will not run if any of the eigenvalues of the correlation matrix are repeated. In addition, if any of the eigenvalues are too close together, depending on the number of pulses integrated, the results will be erroneous due to round-off error in the computer.

The program is written in Fortran IV Language, and utilizes a special computer supplied subroutine to calculate the eigenvalues of the correlation matrix. The accuracy of this subroutine is inversely proportional to the size of the correlation matrix (number of pulses integrated). The program is broken into three distinct parts, as indicated by the position of the comment statements. The first portion is utilized to calculate the bias level, YB in the program, for a given N, number of pulses integrated, and PF, probability of false alarm. YB must lie between ten and two hundred for this portion of the program to converge on a solution. The second portion of the program calculates the correlation matrix when the correlation time constant, TC, the pulse repetition rate, PRR, and RHO, the correlation coefficient resulting from frequency agility, is known. In addition, the eigenvalues of the correlation matrix are calculated in this portion of the program by utilizing the computer supplied subroutine EIGEN. If all of the eigenvalues do not differ by at least 10^{-5} , the program shuts down. The third portion of the program calculates the probability of detection, PSUBD. The partial fraction expansion of

the PDF of the integrator output is utilized in this section of the program.

Unfortunately, the program round-off error caused erroneous outputs for N greater than ten and RHO less than 0.75. However, the basic philosophy for obtaining the solution is described in the program. This same convergence problem was encountered in writing the program for the curves of P_d in Chapter IV. However, a method was finally found which allowed the equation in Chapter IV to converge.

```

C      THIS PROGRAM CALCULATES THE PROBABILITY OF
C      DETECTION FOR A PULSED, NON-COHERENT RADAR
C      WITH FREQUENCY AGILITY. A COMPUTER SUPPLIED
C      SUBROUTINE IS UTILIZED TO CALCULATE THE EIGEN-
C      VALUES OF THE CORRELATION MATRIX. THE TARGET IS
C      ASSUMED TO BE SWERLING'S CASE I, WITH TIME
C      DECORRELATION SUPERIMPOSED. THE FOLLOWING DATA ARE
C      REQUIRED: NUMBER OF PULSES INTEGRATED (N)
C      PROBABILITY OF FALSE ALARM (PF), CORRELATION
C      COEFFICIENT (RHO), PULSE REPETITION RATE (PRR)
C      AND DECORRELATION TIME (TC). N MUST BE LESS THAN
C      FORTY. THE DIMENSION STATEMENT FOR THE F AND R
C      MATRIX MUST CORRESPOND IN SIZE TO N.

```

```

-----
C      DIMENSION F(10,10),R(10)
C      REAL*8 XBAR,X5,S,PSUBD,RORO
C      REAL*8 G(45),C(45),P(45),E(45)
C      READ(1,1000)N,PF,RHO,PRR,TC
1000  FORMAT(I5,E10.3,3F10.5)

```

```

-----
C      PART 1
C      THIS PORTION OF THE PROGRAM FINDS THE BIAS LEVEL
C      (YB) FOR YB BETWEEN 10 AND 200.
C      YB=0.0
C      X1=10.0
C      M=N-1
50  YB=X1+YB
   IF(YB-40.0)60,60,70
60  L1=1
   X2=EXP(-YB)
   GO TO 140
70  IF(YB/2.0-40.0)80,80,90
80  L1=2
   GO TO 135
90  IF(YB/3.0-40.0)100,100,110
100 L1=3
   GO TO 135
110 IF(YB/4.0-40.0)120,120,130
120 L1=4
   GO TO 135
130 L1=5
135 X2=0.0
140 PFT=X2
   DO 160 I=L1,M
   X3=I
   Z=YB*EXP(-YB/X3)
   Z1=1.0
   DO 150 J=1,I
   X4=J
   Z1=Z*X1/X4
150 CONTINUE
   PFT=Z1+PFT
160 CONTINUE
   IF(PFT-PF)170,210,50
170 Z3=ABS(PFT-PF)
   IF(Z3-PF*1.0E-04) 210,210,180
180 YB=YB-X1
   X1=0.1*X1
   GO TO 50

```

```

C -----
C PART 3
C THIS PORTION OF THE PROGRAM FINDS THE PROBABILITY
C OF DETECTION (PSUBD) IF ALL OF THE EIGENVALUES
C ARE DISTINCT.
310 DO 320 I=1,N
    E(I)=F(I,I)
320 CONTINUE
    WRITE(3,1006)N,PF,YB
1006 FORMAT(/1X,'N=',I3,5X,'PF=',E10.3,5X,'YB=',E16.8)
    PRDUC=PRR*TC
    WRITE(3,1007)RHO,PRDUC
1007 FORMAT(/1X,'RHO=',F10.4,5X,'PRODUCT=',F11.4)
    RORO=1.72D+00
    DO 500 L=1,76
        RORO=RORO-0.02D+00
        X5=1.0D+00/RORO

        XBAR=X5*X5*X5*X5
        DO 330 I=1,N
            G(I)=1.0D+00/(1.0D+00+E(I)*XBAR)
330 CONTINUE
            DO 360 I=1,N
                S=1.0D+00
                DO 350 J=1,N
                    IF(J-I)340,350,340
340 S=S*G(J)/(G(J)-G(I))
350 CONTINUE
                C(I)=S
360 CONTINUE
            DO 370 I=1,N
                X5=G(I)*YB
                P(I)=C(I)*DEXP(-X5)
370 CONTINUE
                PSUBD=0.0D+00
                DO 380 I=1,N
                    PSUBD=PSUBD+P(I)
380 CONTINUE
                PD=PSUBD
                PK=ABS(PD)
                IF(PK-0.999)390,390,500
390 IF(PK-0.001)500,400,400
400 WRITE(3,1020)RORO,PD
1020 FORMAT(/1X,'RORO=',D11.4,5X,'PD=',E16.8)
500 CONTINUE
1998 STOP
    END

```

```

C -----
C PART 3
C THIS PORTION OF THE PROGRAM FINDS THE PROBABILITY
C OF DETECTION (PSUBD) IF ALL OF THE EIGENVALUES
C ARE DISTINCT.
310 DO 320 I=1,N
    E(I)=F(I,I)
320 CONTINUE
    WRITE(3,1006)N,PF,YB
1006 FORMAT(/1X,'N=',I3,5X,'PF=',E10.3,5X,'YB=',E16.8)
    PRDUC=PRR*TC
    WRITE(3,1007)RHO,PRDUC
1007 FORMAT(/1X,'RHO=',F10.4,5X,'PRODUCT=',F11.4)
    RORO=1.72D+00
    DO 500 L=1,76
        RORO=RORO-0.02D+00
        X5=1.0D+00/RORO

        XBAR=X5*X5*X5*X5
        DO 330 I=1,N
            G(I)=1.0D+00/(1.0D+00+E(I)*XBAR)
330 CONTINUE
            DO 360 I=1,N
                S=1.0D+00
                DO 350 J=1,N
                    IF(J-I)340,350,340
340 S=S*G(J)/(G(J)-G(I))
350 CONTINUE
                C(I)=S
360 CONTINUE
            DO 370 I=1,N
                X5=G(I)*YB
                P(I)=C(I)*DEXP(-X5)
370 CONTINUE
                PSUBD=0.0D+00
                DO 380 I=1,N
                    PSUBD=PSUBD+P(I)
380 CONTINUE
                PD=PSUBD
                PK=ABS(PD)
                IF(PK-0.999)390,390,500
390 IF(PK-0.001)500,400,400
400 WRITE(3,1020)RORO,PD
1020 FORMAT(/1X,'RORO=',D11.4,5X,'PD=',E16.8)
500 CONTINUE
1998 STOP
    END

```

VITA

Kenneth Lee Horn was born August 28, 1936, in Ada, Oklahoma. His elementary and secondary education was received at various schools in Oklahoma, Kansas, California, Montana, and North Dakota. In September, 1954, he entered the University of Oklahoma from which he graduated with a Bachelor of Science Degree in Electrical Engineering in June, 1962, and a Master of Electrical Engineering Degree in June of 1963. He entered the University of Missouri at Rolla in September, 1966, to pursue his graduate work. His undergraduate education was interrupted three and one half years for a tour of service with the United States Army.

The author was employed by Westinghouse Electric Corporation during the summers of 1960 and 1961. The summer of 1962 was spent in the employ of the University of Oklahoma Research Institute. From March, 1963 to September, 1965, the author was employed by Hughes Aircraft Company. He has been employed by Emerson Electric Company from October 1965, to the present, except for an academic leave of absence from September, 1966 to May, 1968.

He is married to the former Martha Williams and is the father of two sons; Kenneth Sherman and David Lee.